# Polyadic semigroups

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## 1. Introduction

A *polyadic semigroup*, specifically an *n-semigroup*, is an associative $n$-ary operation $F$ on a set $\mathfrak{M}$. I. e., $F$ is a function whose domain is the Cartesian power $\mathfrak{M}^n$ and whose range is a subset of $\mathfrak{M}$, such that each of the $n-1$ equations [1])

$$FFx_1 \dots x_{2n-1} = Fx_1 \dots x_k Fx_{k+1} \dots x_{2n-1} \qquad (1 \leqq k \leqq n-1) \tag{1}$$

holds for every $(2n-1)$-tuple $x_1 \dots x_{2n-1}$ of elements of $\mathfrak{M}$. The main purpose of this note is to prove the following result:

**Theorem 1.** Let $F$ be an $n$-semigroup on a set $\mathfrak{M}$. Assume that $F$ has a *right identity, i. e., there exists an* $(n-1)$-tuple $b_1 b_2 \dots b_{n-2}$ $a$ *(briefly,* $\alpha a$*) such that:*

$$Fxb_1 \dots b_{n-2}a = Fx\alpha a = x \tag{3}$$

*for every $x$ in $\mathfrak{M}$. Define a binary operation $S_\alpha$, and a unary operation $f$ by:*

$$S_\alpha xy = Fx\alpha y, \tag{3}$$

$$fx = Fax\alpha, \tag{4}$$

*for all $x,y$ in $\mathfrak{M}$. Let $c = Fa^n$. Then $S_\alpha$ is a (binary) semigroup with right-identity $a$, $f$ is an endomorphism of $S_\alpha$, and we have*

$$Fx_1x_2 \dots x_n = S_\alpha^n x_1 fx_2 \dots f^{n-1}x_n c \tag{5}$$

*for all $n$-tuples $x_1x_2 \dots x_n$ in $\mathfrak{M}^n$.*

This theorem, which will be proved in § 2, is a generalization of a theorem of M. Hosszú ([2]) for polyadic groups. A polyadic *group (n-group)* [2]) is an $n$-semigroup in which, given any $n$ of the $n+1$ elements $x_1, \dots, x_{n+1}$ of $\mathfrak{M}$, a solution of the equation

$$Fx_1 \dots x_n = x_{n+1} \tag{6}$$

[1]) I use the parentheses-free (Lukasiewicz) functional notation. Expressions such as FF, FFF aa, aaaa will be abbreviated to $F^2$, $F^3$, $a^2$, $a^4$, etc.

[2]) This definition, as well as that of generalized associativity (1), was first introduced by E. Dörnte [1].

for the remaining unknown element of $\mathfrak{M}$ always exists. In other words, each one-place part function [3]) of $F$ is onto $\mathfrak{M}$. This phrasing corresponds to the Huntington definition of a (binary) group as an associative system $G$ where $aG=Ga=G$ ([3]). As in the binary case, it then can be shown that these 1-place part-functions are one-to-one, i.e., the solutions of (6) are unique.

E. L. Post ([4]) in his fundamental Coset Theorem showed that every polyadic group $F$ on a set $\mathfrak{M}$ has a covering (binary) group $K$. This means that there exists a (unique) group $K$ generated by $\mathfrak{M}$, such that for any $n$-tuple $x_1 \dots x_n$ of elements of $\mathfrak{M}$ we have:

$$Fx_1 \dots x_n = K^{n-1}x_1 \dots x_n, \tag{7}$$

and such that $\mathfrak{M}$ is a generating coset of a cyclic factor group of $K$.

The use of the covering group $K$ has the disadvantage of introducing elements not in $\mathfrak{M}$. In [2], Hosszú showed that such introduction can be avoided, and that any polyadic group can be realized as a composite of *a* binary group $G$ on $\mathfrak{M}$ and an automorphism $f$ of $G$. Specifically, he proved that

$$Fx_1 \dots x_n = G^n x_1 fx_2 \dots f^{n-2}x_n c, \tag{8}$$

where $c$ is a fixed element of $\mathfrak{M}$, $fc=c$ and $f^{n-1}x=C^2cxc^{-1}$, i. e., $f^{n-1}$ is an inner automorphism of $G$.[4])

In § 3, we show that Hosszú's theorem is a special case of Theorem 1. Finally, in § 4 we use Theorem 1 to obtain a generalization to $n$-semigroups of a theorem of Dörnte on derivability of $n$-groups.

## 2. Proof of Theorem 1

**Lemma 1.** *The function $S_\alpha$ defined in* (3) *is a (binary) semigroup, i.e.,*

$$S_\alpha S_\alpha xyz = S_\alpha x S_\alpha yz, \tag{9}$$

*for all $x, y, z$ in $\mathfrak{M}$.*

PROOF.
$$\begin{aligned} SSxyz &= FSxy\alpha z = && (3)\\ &= FFx\alpha y\alpha z = && (3)\\ &= Fx\alpha Fy\alpha z = && (1)\\ &= Fx\alpha Syz = && (3)\\ &= SxSyz. && (3)\end{aligned}$$

Note that in this proof we did not have to assume the existence of any identity for $F$. If $F$ has a right-identity $\alpha a$, then it is immediate from (3) that a is a right-identity for $S_\alpha$.

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[3]) A $k$-place part function of F is derived from F by fixing $n$-k of the arguments.

[4]) As a byproduct, Hosszú was able to immediately obtain a theorem of E. Vincze [5], which itself generalized a theorem of J. Aczél on real solutions of the functional equation of associativity.

**Lemma 2.** *The function $f$ defined in* (4) *is an endomorphism of $S_\alpha$, i.e.,*

$$fS_\alpha xy = S_\alpha fxfy, \tag{10}$$

*for all $x$, $y$ in $\mathfrak{M}$.*

PROOF.
$$\begin{aligned} S_\alpha fx\,fy &= S_\alpha Fax\alpha Fay\alpha = && (4)\\ &= F\,Fax\alpha\alpha Fay\alpha = && (3)\\ &= F\,Faxb_1 \dots b_{n-2}\alpha Fay\alpha = &&\\ &= Fa\,Fx\dots Fb_{n-2}\alpha a y\alpha = && (1)\\ &= Fa\,Fxb_1\dots b_{n-2}y\alpha = && (2)\\ &= Fa\,Fx\alpha y\alpha = &&\\ &= f\,Fx\alpha y = && (4)\\ &= fS_\alpha xy. && (3)\end{aligned}$$

**Lemma 3.** *For the functions $f$ and $S_\alpha$ of Theorem 1, we have:*

$$S_\alpha^n x_1 fx_2 \dots f^{n-1}x_n c = S_\alpha^2 x_1 fS_\alpha x_2 \dots fS_\alpha x_{n-1} fx_n c. \tag{11}$$

PROOF. The proof depends on the fact that if $f$ is an endomorphism of $S_\alpha$, then so are all the iterates of $f$. Hence for any positive integer $m$, we have

$$f^m S_\alpha xy = S_\alpha f^m x f^m y. \tag{4a}$$

Now starting with the left hand side of (11), we have:

$$\begin{aligned} S_\alpha^n x_1 fx_2 \dots f^{n-1}x_n c &= S_\alpha^2 x_1 S_\alpha fx_2 \dots S_\alpha f^{n-2}x_{n-1} f^{n-1}x_n c = && (9)\\ &= S_\alpha^2 x_1 S_\alpha fx_2 \dots S_\alpha f^{n-2}x_{n-1} f^{n-2} fx_n c = &&\\ &= S_\alpha^2 x_1 S_\alpha fx_2 \dots f^{n-2} S_\alpha x_{n-1} fx_n c && (4a)\end{aligned}$$

Repeating this procedure leads us, after $(n-2)$ more steps, to (11).

*Remark:* Although $f$ is an endomorphism of $S_\alpha$, $fS_\alpha$ is in general *not*[5]) a semigroup on $\mathfrak{M}$. For example, let $\mathfrak{M}$ be the set of real numbers, and $S$ ordinary real addition. For any $a \neq 0,1$ let $f_a$ denote the linear function defined by $f_a x = a\cdot x$. Then $f_a$ is an endomorphism of $S$. But we have:

$$\begin{aligned} f_a S f_a Sxyz &= a\big(a(x+y)+z\big) = a^2x + a^2y + az \neq \\ &\neq ax + a^2y + a^2z = \big(a(x+a(y+z))\big) = f_a Sx f_a Syz.\end{aligned}$$

A sufficient condition for $fS$ to be associative is that $f$ be idempotent; in this case, if $f$ is onto $\mathfrak{M}$, then $f$ is the identity function on $\mathfrak{M}$.

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[5]) Occasionally one finds the statement such as: If f is an endomorphism of the semigroup S then the image fS is a (sub-) semigroup. This is the result of using the same symbol „S" to denote the set $\mathfrak{M}$ and the semigroup, i.e., the pair ($\mathfrak{M}$, S). What is meant is: The pair (f$\mathfrak{M}$, S′), where S′ is the restriction of S to the set f$\mathfrak{M}$′, is a subsemigroup of ($\mathfrak{M}$, S).

**Lemma 4.** *If* $c = Fa^n$, *then*

$$Fx_1 \dots x_n = S_\alpha^2 x_1 f S_\alpha x_2 \dots f S_\alpha x_{n-1} f x_n c. \tag{12}$$

PROOF.

$$\begin{aligned}
Fx_1 \dots x_n &= FFx_1\alpha a\, Fx_2\alpha a \dots Fx_{n-1}\alpha a\, Fx_n\alpha a = && (2)\\
&= FFx_1\alpha Fax_2\alpha \dots Fax_{n-1}\alpha Fax_n\alpha a = && (1)\\
&= FFx_1\alpha Fax_2\alpha \dots Fax_{n-1}\alpha fx_n a = && (4)\\
&= FFx_1\alpha Fax_2\alpha \dots Fax_{n-1}\alpha F fx_n\alpha a a = && (2)\\
&= FFx_1\alpha Fax_2\alpha \dots FaFx_{n-1}\alpha fx_n\alpha a^2 = && (1)\\
&= FFx_1\alpha Fax_2\alpha \dots FaS_\alpha x_{n-1} fx_n\alpha a^2 = && (3)\\
&= FFx_1\alpha Fax_2\alpha \dots f S_\alpha x_{n-1} fx_n a^2. && (4)
\end{aligned}$$

Applying the procedure in the last 4 steps $n-2$ times in succession we obtain:

$$\begin{aligned}
Fx_1 \dots x_n &= FFx_1\alpha fS_\alpha x_2 \dots f S_\alpha x_{n-1} fx_n a^{n-1} = && (2)\\
&= FFx_1\alpha F fS_\alpha x_2 \dots f S_\alpha x_{n-1} fx_n\alpha a a^{n-1} = && (2)\\
&= FFx_1\alpha fS_\alpha x_2 \dots f S_\alpha x_{n-1} fx_n\alpha Fa^n = && (1)\\
&= FS_\alpha x_1 fS_\alpha x_2 \dots fS_\alpha x_{n-1} fx_n\alpha c = && (2)\\
&= S_\alpha^2 x_1 fS_\alpha x_2 \dots fS_\alpha x_{n-1} fx_n c. && (2)
\end{aligned}$$

Equations (11) and (12) together yield (5), and so *the proof of Theorem* 1 *is complete*.

N. B. Similar arguments will show that a semigroup with a *left* identity $a\alpha$ can be realized in the form:

$$Fx_1 \dots x_n = S_\alpha^n c g^{n-1} x \dots g x_{n-1} x_n,$$

where $S_\alpha xy = Fx\alpha y$, $gx = F\alpha xa$, and $c = Fa^n$.

## 3. Hosszú's Theorem

**Lemma 5.** *If* F *is an* n-*group*, $\alpha$ *is any fixed* $(n-2)$ *tuple, and* $S_\alpha$ *is defined via* (3), *then* $S_\alpha$ *is a group*.

PROOF. By Lemma 1, $S_\alpha$ is a semigroup. And since $F$ is an $n$-group, any equation of the form

$$S_\alpha x_1 x_2 = x_3$$

has a solution whenever two of the $x$'s are given. But this is one of the standard definitions of a group.

In dealing with groups, we shall generally write $G_\alpha$ instead of $S_\alpha$.

**Lemma 6.** *For every* $(n-2)$-*tuple* $\alpha \in \mathfrak{M}^{n-2}$, *there exists an* $a \in \mathfrak{M}$ *such that* $\alpha a$ *and* $a\alpha$ *are each (two-sided) identities of the n-group F.*

PROOF. The (binary) group $G_\alpha$ has an identity, say $a$. Then $G_\alpha ax = Fa\alpha x = x = Gxa = Fx\alpha a$. Thus $\alpha a$ is a right identity and $a\alpha$ a left identity of $F$. Now, to show that $a\alpha$ is also a right identity we have:

$$Fxa\alpha = Fxab_1\dots b_{n-2} = Fxab_1\dots Fb_{n-2}\alpha a = FxFab_1\dots b_{n-2}\alpha a =$$
$$= FxFa\alpha b_1\dots b_{n-2}a = Fxb_1\dots b_{n-2}a = Fx\alpha a = x.$$

Similarly, we can show, that $\alpha a$ is also a left identity.

This lemma is a special case of a more general theorem of POST ([4]): Given any $n-2$ elements $a_1\dots a_{i-1}a_{i+1}\dots a_{n-1}$ of $\mathfrak{M}$ $(i=1, 2, \dots, n-1)$, there exists an element $a_i$ in $\mathfrak{M}$ such that $a_1\dots a_{n-1}$ is an identity of the $n$-group; furthermore, any cyclic permutation of $a_1\dots a_{n-1}$ is also an identity.

**Lemma 7.** *Let* $a\alpha$ *be a right identity of* F *and* $c = Fa^n$. *If* $f$ *is as defined in* (4), *then* $c$ *is a fixed point of* $f$.

PROOF. $fc = Fac\alpha = FaFa^n\alpha = FFaa^n\alpha = FFa^na\alpha = Fca\alpha = c.$

Note that the proof does not require $F$ to be an $n$-group.

**Lemma 8.** *If* F *is an* n*-group, then* f *is an automorphism of* $G_\alpha$ *and* $f^{n-1}$ *an inner automorphism of* $G_\alpha$: *Specifically* $f^{n-1}x = G_\alpha^2\, cxc^{-1}$ *where* $c$ *is as in Lemma* 7.

PROOF. 1. By Lemma 2, $f$ is an endomorphism of $G_\alpha$. Since $F$ is an $n$-group, $f$ is one-to-one onto $\mathfrak{M}$, i. e., an automorphism.

2. $f^{n-1}x = F^{n-1}a^{n-1}x\alpha^{n-1} = F^{n-1}a^{n-2}ax\alpha^{n-1} = F^{n-1}a^{n-2}\,Fa\,a\alpha\, x\alpha^{n-1} =$

$$= FFFa^n\alpha x\alpha F^{n-3}\alpha^{n-2} = FFc\alpha x\alpha F^{n-3}\alpha^{n-2} = FG_\alpha cx\alpha F^{n-3}\alpha^{n-2} = G_\alpha^n cxF^{n-3}\alpha^{n-2}$$

To complete the proof we need only show that $F^{n-3}\alpha^{n-2} = c^{-1}$, i.e., that $G_\alpha cF^{n-3}\alpha^{n-2} = a$. Now

$$GcF^{n-3}\alpha^{n-2} = G\,Fa^n\,F^{n-3}\alpha^{n-2} = FF\,a^n\alpha F^{n-3}\alpha^{n-2} =$$
$$= F^{n-1}a^n\alpha^{n-1} = F^{n-2}a^{n-1}\,Fa\,\alpha b_1\,\dots\,b_{n-2}\alpha^{n-3},$$

where $b_1\dots b_{n-2} = \alpha$. Since $a\alpha$ is a left identity, we have

$$GcF^{n-3}\alpha^{n-2} = F^{n-2}a^{n-1}b_1\,\dots\,b_{n-2}\alpha^{n-3} = F^{n-2}a^{n-1}\alpha^{n-2}.$$

Repeating the procedure, we reduce the right hand side to $Faa\alpha = a$.

Lemmas 5, 6, 7 and 8 together show that Hosszú's Theorem is a special case of Theorem 1.

## 4. Derivable n-semigroups

If $Fa^n = a$, we call $a$ an *idempotent* element of $F$.

**Lemma 9.** *Let* $F$ *be an* $n$*-semigroup with an idempotent element* $a$, *and let a function* $f$ *be defined by*

(13) $$fx = Faxa^{n-2}.$$

*Then* $f^{n-1}x = F^2a^{n-1}xa^{n-1}$, $f^n = f$ *and* $f^{n-1}$ *is an idempotent function, i.e.,* $f^{n-1}f^{n-1} = f^{n-1}$.

PROOF 1. $f^{n-1}x = F^{n-1}a^{n-1}xa^{(n-2)(n-1)} = F^{n-1}a^{n-1}xa^n a^{(n-2)(n-1)-n} =$

$$= F^{n-2}a^{n-1}x\,Fa^n a^{(n-2)(n-1)-n} = F^{n-2}a^{n-1}xaa^{(n-2)(n-1)-n} = F^{n-2}a^{n-1}xa^{(n-3)(n-1)}.$$

Repeating the above steps $n-2$ more times we get:

$$f^{n-1}x = F^2a^{n-1}xa^{n-1}.$$

2. $f^n x = f^{n-1}fx = F^2a^{n-1}fx\,a^{n-1} = F^2a^{n-1}\,Fa\,xa^{n-2}a^{n-1} =$

$$= F^2a^n x\,Fa^n a^{n-3} = Fa\,xaa^{n-3} = Fa\,xa^{n-2} = fx.$$

3. $f^{n-1}f^{n-1} = f^{2n-2} = f^n f^{n-2} = ff^{n-2} = f^{n-1}$.

**Lemma 10.** *Let $F$ be an n-semigroup on a set $\mathfrak{M}$. Let $a$ be an element of $\mathfrak{M}$, and let $f$ be as defined in* (13). *Then the following statements are equivalent:* (i) *$a$ is an idempotent element of $F$, and $f$ is onto $\mathfrak{M}$;* (ii) *$a^{n-1}$ is a (2-sided) identity of $F$.*

PROOF. (i) implies (ii).

Since $f$ is onto $\mathfrak{M}$, $f^{n-1}$ is onto $\mathfrak{M}$. But, by Lemma 9, $f^{n-1}$ is idempotent, and an idempotent function onto a set is the identity function. We therefore have:

$$x = f^{n-1}x = F^2a^{n-1}xa^{n-1} = F^2aa^{n-2}xa^{n-1} = F^2\,Fa^n a^{n-2}xa^{n-1} =$$
$$= F^2a^{n-1}(Fa^{n-1}x)a^{n-1} = f^{n-1}\,Fa^{n-1}x = Fa^{n-1}x,$$

i.e., $a^{n-1}$ is a left identity of $F$. Similarly we can show that $a^{n-1}$ is a right identity.

(ii) implies (i).

1. $Fa^n = Faa^{n-1} = a$ i.e., a is idempotent.
2. $f^{n-1}x = F^2a^{n-1}xa^{n-1} = Fa^{n-1}\,Fx\,a^{n-1} = Fa^{n-1}x = x$,

i.e., $f^{n-1}$ is onto $\mathfrak{M}$. Since the range of $f^{n-1}$ is a subset of the range of $f$, the proof is complete.

DÖRNTE ([1]) already distinguished between genuine (echte) and derivable (ableitbare) $n$-groups. An $n$-semigroup $F$ *is derivable from a binary semigroup $S$* if $F = S^{n-1}$.

**Theorem 2.** *Let* F *be an* n-*semigroup on a set $\mathfrak{M}$. Then the following statements are equivalent:* (i) *$F$ is derivable from a (binary) semigroup $S$ on $\mathfrak{M}$ with identity $a$;* (ii) *$F$ has an idempotent element $a$ such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$;* (iii) *there is an element $a \in \mathfrak{M}$ such that $a^{n-1}$ is a right identity of $F$, and such that $Faxa^{n-2} = x$ for all $x \in \mathfrak{M}$.*

PROOF. (i) implies (ii). $Fa^n = S^{n-1}a^n = a$, i.e., $a$ is an idempotent element of $F$. Furthermore

$$Fa\,xa^{n-2} = S^{n-1}axa^{n-2} = x.$$

(ii) implies (iii). Since $fx = Faxa^{n-2} = x$ means that $f$ is onto $\mathfrak{M}$, this implication follows from Lemma 10.

(iii) implies (i). Let $\alpha = a^{n-2}$. By hypothesis $\alpha a$ is a (right) identity. We can therefore apply Theorem 1 and obtain

$$Fx_1 \dots x_n = S_\alpha^n x_1 fx_2 \dots f^{n-1} x_n Fa^n = S_\alpha^n x_1 fx_2 \dots f^{n-1} x_n a.$$

Now $fx = Fax\alpha = Faxa^{n-2} = x$ by hypothesis, and $a$ is the identity of $S_\alpha$. Hence we have

$$Fx_1 \dots x_n = S_\alpha^{n-1} x_1 \dots x_n.$$

Since any group has an identity, Theorem 2 includes as a special case the necessary and sufficient criterion for $n$-groups to be derivable from binary groups (Dörnte, [1], § 2, Theorem 6).

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Added in proof. I am indebted to Professor B. M. Schein for directing my attention to the fundamental work of L. M. Gluskin on $n$-semigroups and related systems (Positional Operatives, Doklady Akad. Nauk SSSR **157**, 1964, 767—770; Positional Operatives, Mat. Sbornik **68**, 1965, 444—472). In particular, Theorem 5 in the Doklady paper is essentially a weaker form of Theorem 1 of this paper.

## Bibliography

[1] W. Dörnte, Untersuchungen über einen verallgemeinerten Gruppenbegriff. *Math. Z.* **29** (1928), 1—19.

[2] M. Hosszú, On the explicit form of $n$-group operations. *Publ. Math. Debrecen* **10** (1963), 88—92.

[3] E. V. Huntington, Simplified definition of a group. *Bull. Amer. Math. Soc.* **8** (1901—02), 296—300.

[4] E. L. Post, Polyadic Groups, *Trans. Amer. Math. Soc.* **48** (1940), 208—350.

[5] E. Vincze, Verallgemeinerung eines Satzes über assoziative Funktionen von mehreren Veränderlichen. *Publ. Math. Debrecen* **8** (1961), 68—74.

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