

Radius of convexity of a certain class of meromorphically starlike functions.

By R. M. GOEL (Patiala)

1. *Introduction:* Let S denote the class of all univalent functions $f(z)$ given by

$$(1) \quad f(z) = \frac{1}{z} + b_0 + b_1 z + \dots,$$

regular and starlike of order α for $0 < |z| < 1$, that is $f(z)$ satisfies the condition

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha, \quad 0 \leq \alpha < 1.$$

M. S. ROBERTSON ([1]) determined the radius of convexity for the class S for $\alpha = 0, \frac{1}{2}$.

In a recent paper [2] K. S. PADAMNABHAN solved the problem for the case $\frac{1}{2} \leq \alpha < 1$. His method fails to give a sharp result for $0 \leq \alpha < \frac{1}{2}$.

In this short paper we shall obtain the radius of convexity for the class S for $0 \leq \alpha \leq \frac{1}{2}$.

2. We shall need the following lemma:

Lemma I. *Let $H(z)$ be regular and satisfy $\operatorname{Re} \{H(z)\} > \alpha$, $0 \leq \alpha < 1$, for $|z| < 1$ and let $H(0) = 1$. Then we have*

$$H(z) = \frac{1 - (2\alpha - 1)w(z)}{1 - w(z)},$$

where $w(z)$ is any regular function satisfying $w(0) = 0$, $|w(z)| \leq |z|$ for $|z| < 1$, and any function $H(z)$ given by the above formula is regular and satisfies $\operatorname{Re} \{H(z)\} > \alpha$ in $|z| < 1$.

The proof can be found in [4], p. 99.

We prove the following

Theorem 1. *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \dots$$

be regular and starlike of order α , $0 \leq \alpha \leq \frac{1}{2}$, for $0 < |z| < 1$. Then $f(z)$ maps

$$|z| < \left\{ \frac{2\sqrt{1-\alpha^2} - (1+\alpha)}{3-5\alpha} \right\}^{\frac{1}{2}}$$

on to a domain the complement of which is convex. The estimate is sharp.

PROOF. Since $f(z)$ is starlike of order α , therefore

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} < -\alpha, \quad \text{for } 0 < |z| < 1,$$

and hence $f(z)$ does not vanish in $|z| < 1$. The function $-\frac{zf'(z)}{f(z)}$ is regular in $|z| < 1$ and satisfies the conditions of lemma. Therefore, we can write

$$(2) \quad \frac{zf'(z)}{f(z)} = -\frac{1 - (2\alpha - 1)w(z)}{1 - w(z)},$$

where $w(z)$ is regular for $|z| < 1$ and satisfies $w(0) = 0$, $|w(z)| \leq |z|$.

By differentiating and simplifying (2), we have

$$(3) \quad -\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{[1 + (1 - 2\alpha)w(z)]^2 - 2(1 - \alpha)zw'(z)}{[1 - w(z)][1 + (1 - 2\alpha)w(z)]} = K(z), \quad \text{say.}$$

We wish to find the largest circle with centre at the origin with in which $f(z)$ satisfies the inequality.

$$(4) \quad \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \leq 0,$$

that is

$$(5) \quad \operatorname{Re} \{K(z)\} \geq 0,$$

where $K(z)$ is given by (3). In turn, the inequality (5) is equivalent to

$$(6) \quad \left| \frac{K(z) - 1}{K(z) + 1} \right| = \left| \frac{(1 - \alpha)[(1 - 2\alpha)(w(z))^2 + (w(z) - zw'(z))]}{1 - 2\alpha w(z) + (1 - \alpha)(w(z) - zw'(z)) - \alpha(1 - 2\alpha)(w(z))^2} \right| \leq 1 \\ \equiv \frac{(1 - \alpha)[(1 - 2\alpha)|w(z)|^2 + |w(z) - zw'(z)|]}{1 - 2\alpha|w(z)| - (1 - \alpha)|w(z) - zw'(z)| - \alpha(1 - 2\alpha)|w(z)|^2} \leq 1.$$

This will be satisfied if

$$(7) \quad (1 - 2\alpha)|w(z)|^2 + 2(1 - \alpha)|w(z) - zw'(z)| + 2\alpha|w(z)| - 1 \leq 0.$$

Let

$$(8) \quad \varphi(z) = \frac{w(z)}{z},$$

then $\varphi(z)$ is regular and satisfies $|\varphi(z)| \leq 1$, for $|z| < 1$. Also for such functions $\varphi(z)$ we have ([3], p. 18),

$$(9) \quad |\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$

We shall show that the estimate (9) is sharp.

Take

$$\varphi(z) = -\frac{\gamma - z}{1 - \bar{\gamma}z}, \quad |\gamma| < 1.$$

Then

$$\varphi'(z) = \frac{1 - |\gamma|^2}{(1 - \bar{\gamma}z)^2}.$$

$$(10) \quad |\varphi'(z)| = \frac{|1 - \bar{\gamma}z|^2 - |\gamma - z|^2}{(1 - |z|^2)|1 - \bar{\gamma}z|^2} = \frac{1 - |\varphi(z)|^2}{1 - |z|^2}.$$

Differentiating (8) we get

$$(11) \quad \varphi'(z) = \frac{zw'(z) - w(z)}{z^2}$$

From (10) and (11) we obtain

$$(12) \quad |zw'(z) - w(z)| = \frac{|z|^2 - |w(z)|^2}{1 - |z|^2}.$$

Consequently (7) becomes

$$(1 - 2\alpha)|\omega(z)|^2 + 2(1 - \alpha)\frac{r^2 - |\omega(z)|^2}{1 - r^2} + 2\alpha|\omega'(z)| - 1 \geq 0,$$

here $r = |z|$, or

$$|w(z)|^2[1 + (1 - 2\alpha)r^2] - 2\alpha(1 - r^2)|w(z)| + (2\alpha - 3)r^2 + 1 \geq 0,$$

or

$$[1 + (1 - 2\alpha)r^2]\left[|\omega(z)| - \frac{\alpha(1 - r^2)}{1 + r^2(1 - 2\alpha)}\right]^2 + \left[(2\alpha - 3)r^2 + 1 - \frac{\alpha^2(1 - r^2)^2}{1 + (1 - 2\alpha)r^2}\right] \geq 0.$$

This last inequality is satisfied if

$$(2\alpha - 3)r^2 + 1 - \frac{\alpha^2(1 - r^2)^2}{1 + (1 - 2\alpha)r^2} \geq 0,$$

or if

$$(3 - 5\alpha)r^4 + 2(1 + \alpha)r^2 - (1 + \alpha) \geq 0,$$

which holds for

$$(13) \quad r^2 < \frac{2\sqrt{1 - \alpha^2} - (1 + \alpha)}{3 - 5\alpha} = b^2, \quad \text{say.}$$

Therefore, $f(z)$ is convex in $0 < |z| < b$.

Now we shall show that the estimate (13) is sharp. Let

$$w(z) = -\frac{z(z - \gamma)}{1 - \gamma z},$$

where γ is defined by

$$(14) \quad \gamma = \frac{1 - b^2(4\alpha - 1)}{4b(1 - \alpha)} = \frac{-7b^6 - 5b^4 + 3b^2 - 1}{8b(b^2 - 1)^2}.$$

It is easy to show that $|\gamma| < 1$. Then the function $w(z) = -\frac{z(z-\gamma)}{1-\gamma z}$ satisfies (12) and makes

$$[1 + (1 - 2\alpha)w(z)]^2 - 2(1 - \alpha)zw'(z) = 0, \quad \text{for } z = b,$$

which implies that $K(b) = 0$. It follows that the function $f(z)$ defined by

$$\frac{zf'(z)}{f(z)} = -\frac{1 - (2\alpha - 1)w(z)}{1 - w(z)}, \quad w(z) = \frac{-z(z-\gamma)}{1-\gamma z}$$

is not convex in $0 < |z| < R$ if R exceeds b . Therefore the bound b is sharp.

Acknowledgement: I wish to thank Professor VIKRAMADITYA SINGH for his kind help and guidance.

References

- [1] M. S. ROBERTSON, Some radius of convexity problems, *Michigan Math. J.* **10** (1963) 231—236.
- [2] K. S. PADAMNABHAN, On the radius of convexity of certain class of meromorphically starlike functions in the unit circle, *Math. Z.* **91** (66), 308—313.
- [3] C. CARATHEODORY, Theory of functions, Vol. 2, *New York*, 1954.
- [4] K. S. PADAMNABHAN, On certain classes of meromorphic functions in the unit circle, *Math. Z.* **89** (1965), 98—107.

(Received August 22, 1966.)