

Sets of radical classes

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1. Introduction

A class $\mathcal{C}$ of rings is said to be *homomorphically closed* if $R \in \mathcal{C}$ implies $R\Phi \in \mathcal{C}$ for every homomorphism Φ of R , and is said to be *hereditary* if $R \in \mathcal{C}$ implies $I \in \mathcal{C}$ for every ideal I of R . A class which is both homomorphically closed and hereditary is said to be a *universal* class. In the following all rings considered are from an arbitrary but fixed class $\mathcal{C}$ of (not necessarily associative) rings.

For a class $\mathcal{P} \subseteq \mathcal{C}$ define

$$(1) \quad \mathcal{SP} = \{R \in \mathcal{C} \mid I \notin \mathcal{P} \text{ for every non-zero ideal } I \text{ of } R\}$$

and

$$(2) \quad \mathcal{UP} = \{R \in \mathcal{C} \mid R\Phi \notin \mathcal{P} \text{ for every non-zero homomorphism } \Phi \text{ of } R\}$$

Remark 1. It is well-known that if $\mathcal{P}$ is a radical class [1, 3] then $\mathcal{SP}$ is its semisimple class and $\mathcal{USP} = \mathcal{P}$ ([4] Theorem 1, p. 4). (Note that the proofs of [4] Chapter I apply equally well to classes of possibly non-associative rings.)

Remark 2. On the other hand, if $\mathcal{P}$ is a semisimple class then $\mathcal{UP}$ is a radical class (called the *upper* radical defined by the class $\mathcal{P}$) and $\mathcal{SPUP} = \mathcal{P}$ ([4] Theorem 2, p. 5).

Remark 3. If $\mathcal{P}$ is a radical class we will write $\mathcal{P}(R)$ for the $\mathcal{P}$ -radical of the ring R . Note that if H is an ideal of R such that $R/H \in \mathcal{SP}$ then $\mathcal{P}(R) \subseteq H$. This is clear since otherwise $(\mathcal{P}(R) + H)/H \cong \mathcal{P}(R)/(\mathcal{P}(R) \cap H)$ would be a non-zero $\mathcal{P}$ -ideal of R/H contradicting the definition (1).

Remark 4. In the Kurosh construction of the *lower* radical for a homomorphically closed class of rings $\mathcal{H}$ (as modified by ANDERSON, DIVINSKY and SULINSKI ([4] footnote p. 12) a class $\mathcal{H}_\alpha$ is of degree α over $\mathcal{H} = \mathcal{H}_1$ (for an arbitrary ordinal $\alpha > 1$) if $\mathcal{H}_\alpha = \{R \in \mathcal{C} \mid \text{every non-zero homomorphic image } R\Phi \text{ has an ideal } I \in \mathcal{H}_\beta \text{ for some } \beta < \alpha\}$. We will write $\mathcal{LH}$ for the lower radical class defined by $\mathcal{H}$, namely $\mathcal{LH} = \bigcup_{\alpha} \mathcal{H}_\alpha$. Note that from the definitions (1) and (2) it follows that $\mathcal{H}_2 = \mathcal{USH}$, for if every $R\Phi$ has an ideal $I \in \mathcal{H}_1 = \mathcal{H}$ then $R\Phi \notin \mathcal{SH}$, so that $R \in \mathcal{USH}$. The converse is also clear.

2. The intersection radical

Let $\{\mathcal{P}_j\}_{j \in J}$ be a set of radical subclasses of $\mathcal{C}$.

Theorem 1. If $\mathcal{P} = \bigcap_{j \in J} \mathcal{P}_j$ then $\mathcal{P}$ is a radical class such that for any ring R

$$(3) \quad \mathcal{P}(R) \subseteq \bigcap_{j \in J} \mathcal{P}_j(R).$$

If all the classes $\mathcal{P}_j$ are hereditary, then so is $\mathcal{P}$, and the relation (3) becomes an equality.

PROOF. Since each $\mathcal{P}_j$ is homomorphically closed, the same is true of $\mathcal{P}$. Also if $R \notin \mathcal{P}$ then $R \notin \mathcal{P}_j$ for some j , so there exists a non-zero $R\Phi \in \mathcal{P}_j \subseteq \mathcal{P}$. By [4] (Theorem 1, p. 4) $\mathcal{P}$ is thus a radical class. Relation (3) is clear from the definition of the $\mathcal{P}$ -radical. It is also clear that when all $\mathcal{P}_j$ are hereditary then so is $\mathcal{P}$, and if $H = \bigcap_{j \in J} \mathcal{P}_j(R)$ then H is an ideal of each $\mathcal{P}_j(R)$. By the hereditary property, $H \in \mathcal{P}_j$ whence $H \in \mathcal{P}$ so that $H \subseteq \mathcal{P}(R)$.

Call $\mathcal{P}(R)$ the *intersection radical* of the set $\{\mathcal{P}_j\}$. We now proceed to construct for an arbitrary ring R a collection of subrings which, when R is associative, all equal $\mathcal{P}(R)$.

Define $V_1 = R$. For a given ordinal β assume that subrings V_α have been defined for all ordinals $\alpha < \beta$ and define:

$$(4) \quad V_\beta = \begin{cases} \bigcap_{\alpha < \beta} V_\alpha & \text{if } \beta \text{ is a limit ordinal.} \\ \mathcal{P}_j(V_{\beta-1}) & \text{for some } j \in J \text{ (if such exists) for which } \mathcal{P}_j(V_{\beta-1}) \neq V_{\beta-1}. \end{cases}$$

Since $\{V_\alpha\}$ is a set, it cannot be inductive for the class of all ordinals. Hence there must exist an ordinal γ such that $\mathcal{P}_j(V_\gamma) = V_\gamma$ for all $j \in J$.

Note that (4) may permit considerable arbitrariness at each non-limit ordinal. We have thus in general defined a collection of such subrings V_γ . As the following theorem shows, however, these coincide in the case R is associative.

Theorem 2. For an arbitrary ring R let V_β be defined by (4). Then there exists an ordinal γ such that $\mathcal{P}(V_\gamma) = V_\gamma$. In all cases $\mathcal{P}(R) \subseteq V_\gamma$, and if R is associative this becomes an equality.

PROOF. As in the above remarks there exists γ such that $\mathcal{P}_j(V_\gamma) = V_\gamma$ for all $j \in J$. But then $V_\gamma \in \bigcap_{j \in J} \mathcal{P}_j$ so that $V_\gamma \subseteq \mathcal{P}(V_\gamma)$ and we have equality.

Since $V_1 = R$, $\mathcal{P}(R) \subseteq V_1$. Thus assume for induction that for a given ordinal β , $\mathcal{P}(R) \subseteq V_\alpha$ for all ordinals $\alpha < \beta$. Then clearly by definition $\mathcal{P}(R) \subseteq V_\beta$ when β is a limit ordinal. If $\beta - 1$ exists then by the induction hypothesis, $\mathcal{P}(R) \subseteq V_{\beta-1}$ and since $\mathcal{P}(R) \in \mathcal{P}_j$ it is a $\mathcal{P}_j$ -ideal of $V_{\beta-1}$. Thus $\mathcal{P}(R) \subseteq \mathcal{P}_j(V_{\beta-1}) = V_\beta$. By induction $\mathcal{P}(R) \subseteq V_\beta$ for all β and hence $\mathcal{P}(R) \subseteq V_\gamma$. But we already have that V_γ is a $\mathcal{P}$ -ring, so the reverse inequality would follow if V_γ were an ideal of R . Now when R is associative, the radical of an ideal is an ideal of R [4; Theorem 47, p. 124], and so by induction V_γ is an ideal of R .

3. The join radical

Again let $\{\mathcal{P}_j\}_{j \in J}$ be a set of radical classes and write $\mathcal{H} = \bigcup_{j \in J} \mathcal{P}_j$. In general $\mathcal{H}$ is not a radical class, but in the associative case turns out to be only one degree from a radical. First we have

Lemma 1. $\mathcal{S}\mathcal{H} = \bigcap_{j \in J} \mathcal{S}\mathcal{P}_j$ and $\mathcal{H} \subseteq \mathcal{U}\mathcal{S}\mathcal{H}$.

PROOF. The first statement is clear since a ring has no non-zero ideals in any $\mathcal{P}_j$ if and only if it is a member of every $\mathcal{S}\mathcal{P}_j$. For the second relation it suffices, by remark 4, to observe that $\mathcal{H}$ is homomorphically closed.

We will write $\mathcal{J}(R)$ for the sum of all ideals of a ring R which are members of $\mathcal{U}\mathcal{S}\mathcal{H}$. From lemma 1 it follows that $\Sigma \mathcal{P}_j(R) \subseteq \mathcal{J}(R)$.

Let $\mathcal{A}$ be the class of all associative rings and write $\bar{\mathcal{S}}\mathcal{H} = \mathcal{S}\mathcal{H} \cap \mathcal{A}$. (Note that this is equivalent to $\mathcal{S}\mathcal{H}$ defined by (1) relative to the universal class $\bar{\mathcal{C}} = \mathcal{C} \cap \mathcal{A}$.)

Theorem 3. $\bar{\mathcal{S}}\mathcal{H}$ is a semisimple class in $\bar{\mathcal{C}}$ and hence for R an associative ring $\mathcal{J}(R) = \mathcal{L}\mathcal{H}(R)$.

PROOF. First suppose $R \in \bar{\mathcal{S}}\mathcal{P}_j$ for all $j \in J$. Since $\bar{\mathcal{S}}\mathcal{P}_j$ is a semisimple class for a radical class $(\mathcal{P}_j \cap \mathcal{A})$ in an associative universal class $\bar{\mathcal{C}}$, it is a hereditary class [4; Corollary 2, p. 125]. Thus every ideal of R is a member of $\bigcap_{j \in J} \bar{\mathcal{S}}\mathcal{P}_j$, so by Lemma 1 it follows that $\bar{\mathcal{S}}\mathcal{H}$ is hereditary. On the other hand, let $R \in \bar{\mathcal{C}}$ such that $R \notin \bar{\mathcal{S}}\mathcal{P}_j$ for some j . Then ([4] Theorem 2, p. 5) R has a non-zero ideal I all of whose non-zero images $I\Phi \notin \bar{\mathcal{S}}\mathcal{P}_j$. Thus $I\Phi \notin \bigcap_{j \in J} \bar{\mathcal{S}}\mathcal{P}_j = \bar{\mathcal{S}}\mathcal{H}$ for all homomorphisms Φ . Hence, again by [4; Theorem 2], $\bar{\mathcal{S}}\mathcal{H}$ is a semisimple subclass of $\bar{\mathcal{C}}$.

Thus the upper radical $\bar{\mathcal{U}}\bar{\mathcal{S}}\mathcal{H}$ defined in $\bar{\mathcal{C}}$ is a radical class with radical $\mathcal{J}(R)$ in any $R \in \bar{\mathcal{C}}$. But (relative to $\bar{\mathcal{C}}$) $\mathcal{H}_2 = \bar{\mathcal{U}}\bar{\mathcal{S}}\mathcal{H} \subseteq \mathcal{L}\mathcal{H}$ and by the minimality of the lower radical ([4] lemma 5, p. 13) this becomes an equality. Thus for any $R \in \bar{\mathcal{C}}$ we have $\mathcal{J}(R) = \mathcal{L}\mathcal{H}(R)$. We will call $\mathcal{J}(R)$ the *join radical* for the set $\{\mathcal{P}_j\}$.

We now proceed, by a construction analogous to that of Section 2, to construct a collection of ideals which, when R is associative, will all coincide with $\mathcal{J}(R)$.

Let $W_1 = 0$ and for a given ordinal β assume that ideals W_α have been defined for all ordinals $\alpha < \beta$. Then define W_β as follows:

(5) $W_\beta = \bigcup_{\alpha < \beta} W_\alpha$ when β is a limit ordinal. Otherwise by

$$W_\beta / W_{\beta-1} = \mathcal{P}_j(R / W_{\beta-1}) \text{ for some } j \in J \text{ (if such exists) for which } \mathcal{P}_j(R / W_{\beta-1}) \neq 0.$$

This means, again, that to avoid 1-1 correspondence between the class of all ordinals and a subset of the set of all ideals of R , there must exist some ordinal γ such that

$$(6) \quad \mathcal{P}_j(R / W_\gamma) = 0 \text{ for all } j \in J.$$

Note that here also the arbitrariness of the choice of j at each non-limit ordinal means that we have in general defined a collection of such W_γ .

Theorem 4. *For a ring R let W_γ be an ideal defined by (5), satisfying condition (6). Then $W_\gamma \subseteq \mathcal{J}(R)$ and if R is associative this becomes an equality.*

PROOF. Suppose for induction we assume for a given ordinal β that $W_\alpha \in \mathcal{US}\mathcal{H}$ for all $\alpha < \beta$. If β is a limit ordinal, W_β also satisfies this condition. So suppose $W_\beta/W_{\beta-1} = \mathcal{P}_j(R/W_{\beta-1})$ for some $j \in J$. Let $W_\beta/K \neq 0$ be an arbitrary homomorphic image of W_β . Since $W_1 = 0 \subseteq K$ while $W_\beta \not\subseteq K$ we may assume there exists an ordinal α such that $W_\alpha \subseteq K$ but $W_{\alpha+1} \not\subseteq K$. Thus $(W_{\alpha+1} + K)/K \cong W_{\alpha+1}/W_{\alpha+1} \cap K \neq 0$. But $W_\alpha \subseteq K$ so from the natural homomorphism of $W_{\alpha+1}/W_\alpha$ onto $W_{\alpha+1}/W_{\alpha+1} \cap K$ and the fact that by (5) $W_{\alpha+1}/W_\alpha \in$ some $\mathcal{P}_j$, it follows that $(W_{\alpha+1} + K)/K \in \mathcal{P}_j$. Thus W_β/K contains a non-zero ideal from $\mathcal{H}$. Hence $W_\beta/K \notin \mathcal{US}\mathcal{H}$ and since K was arbitrary it follows that $W_\beta \in \mathcal{US}\mathcal{H}$. By induction all W_β and hence $W_\gamma \in \mathcal{US}\mathcal{H}$ so that $W_\gamma \subseteq \mathcal{J}(R)$.

Now by condition (6) we have $R/W_\gamma \in \bigcap_{j \in J} \mathcal{SP}_j = \mathcal{SH}$. By Theorem 3 the class $\overline{\mathcal{SH}}$ is semisimple in $\overline{\mathcal{C}}$ and is thus the semisimple class of the radical class $\overline{\mathcal{US}\mathcal{H}}$. From Remark 3 it follows that when R is associative $\mathcal{J}(R) \subseteq W_\gamma$ and equality follows.

References

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