

BV-solutions of some systems of nonlinear functional equations

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Abstract. In the paper the solutions of bounded variation (BV-solutions) of the systems of functional equations $\varphi_i(x) = h_i(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_m[f_{i,m}(x)])$ and $\varphi_i[f(x)] = g_i(x, \varphi_1(x), \dots, \varphi_m(x))$, $i = 1, 2, \dots, m$, are considered. Under suitable hypotheses about given functions h_i , g_i and $f_{i,j}$ it is proved that the first system has in an interval $\langle a, b \rangle$ a unique BV-solutions and that BV-solution of the second system in an interval $\langle a, b \rangle$ depends on an arbitrary function.

Introduction. In this paper we consider the solutions of bounded variation (BV-solutions) of the systems of functional equations

$$(1) \quad \varphi_i(x) = h_i(x, \varphi_i[f_{i,1}(x)], \dots, \varphi_m[f_{i,m}(x)]), \quad i = 1, 2, \dots, m$$

and

$$(2) \quad \varphi_i[f(x)] = g_i(x, \varphi_1(x), \dots, \varphi_m(x)), \quad i = 1, 2, \dots, m,$$

where φ_i are any unknown functions.

In our previous papers [1], [2] (with J. MATKOWSKI) and [3] we have considered BV-solutions of the single functional equations (linear, nonlinear of first order and nonlinear of higher order). This is the first paper in which the BV-solutions of the systems of functional equations are considered. The special role in our considerations play some Lemmas and Theorems proved by Professor J. MATKOWSKI in his habilitation work [4].

Let (X, d) be a metric space and $I \subset \mathbb{R}$ an interval. By $P(I)$ we denote the set of all finite partitions $p : x_0 < x_1 < \dots < x_s$ of the

interval I , where $x_i \in I$, $i = 0, 1, \dots, s$. Similarly as in the papers [2] and [3], for the function $\varphi : I \rightarrow M$ we denote by

$$(3) \quad \text{Var}_I \varphi := \sup_{P(I)} \sum_{i=1}^s d(\varphi(x_i), \varphi(x_{i-1})).$$

By $\text{BV}(I)$ we denote the space of all functions $\varphi : I \rightarrow X$ such that $\text{Var}_I \varphi < \infty$.

Also similarly as in [4] let us define the numbers $c_{i,k}^{(r)}$ as follows:

$$(4) \quad c_{i,k}^{(r+1)} = \begin{cases} c_{1,1}^{(r)} c_{i+1,k+1}^{(r)} - c_{i+1,1}^{(r)} c_{1,k+1}^{(r)} & \text{for } i = k \\ c_{1,1}^{(r)} c_{i+1,k+1}^{(r)} + c_{i+1,1}^{(r)} c_{1,k+1}^{(r)} & \text{for } i \neq k \end{cases}$$

$$i, k = 1, 2, \dots, m - r - 1; \quad r = 0, 1, \dots, m - 2.$$

1. In this section we assume the following hypotheses:

- (i) (X, d) is a complete metric space,
- (ii) $f_{i,k} : \langle a, b \rangle \rightarrow \langle a, b \rangle$ are continuous and strictly increasing in $\langle a, b \rangle$, $i, k = 1, 2, \dots, m$,
- (iii) $h_i : \langle a, b \rangle \times X^m \rightarrow X$, $i = 1, 2, \dots, m$,
- (iv) There are $H_i \in \text{BV} \langle a, b \rangle$ and $\ell_{i,k} \in \mathbb{R}$ such that $0 < \ell_{i,k} < 1$, $i, k = 1, 2, \dots, m$, and

$$(5) \quad \begin{aligned} & d(h_i(x, y_1, \dots, y_m), h_i(\bar{x}, \bar{y} - 1, \dots, \bar{y}_m)) \\ & \leq \sum_{k=1}^m \ell_{i,k} d(y_k, \bar{y}_k) + d(H_i(x), H_i(\bar{x})), \\ & i = 1, 2, \dots, m, \quad (x, y_1, \dots, y_m), (\bar{x}, \bar{y}_1, \dots, \bar{y}_m) \in \langle a, b \rangle \times X^m \end{aligned}$$

and the numbers

$$(6) \quad c_{i,k}^{(0)} = \begin{cases} 1 - \ell_{i,k} & \text{for } i = k \\ \ell_{i,k} & \text{for } i \neq k \end{cases} \quad i, k = 1, 2, \dots, m$$

fulfil the conditions

$$(7) \quad c_{i,i}^{(r)} > 0, \quad i = 1, 2, \dots, m - r; \quad r = 0, 1, \dots, m - 1,$$

where $c_{i,k}^{(r)}$ are defined by (4).

Now we have the following:

Theorem 1. *If hypotheses (i)-(iv) are fulfilled then the system of equations (1) has in the interval $\langle a, b \rangle$ a unique solution $\varphi = (\varphi_1, \dots, \varphi_m)$, such that $\varphi_i \in \text{BV} \langle a, b \rangle$, $i = 1, 2, \dots, m$. This solution is given by the formula $\varphi_i = \lim_{n \rightarrow \infty} \varphi_{i,n}$ (uniformly in $\langle a, b \rangle$), where $\varphi_{i,n+1}(x) = h_i(x, \varphi_{1,n}[f_{i,1}(x)], \dots, \varphi_{m,n}[f_{i,m}(x)])$, $x \in \langle a, b \rangle$, $i = 1, 2, \dots, m$, $n = 0, 1, \dots$ and $\varphi_{i,0} \in \text{BV} \langle a, b \rangle$ is arbitrarily choosen. Moreover*

$$(8) \quad \text{Var}_{\langle a, b \rangle} \varphi_i \leq \frac{\sum_{\substack{k=1 \\ k \neq i}}^m \ell_{i,k} \text{Var}_{\langle a, b \rangle} \varphi_k + \text{Var}_{\langle a, b \rangle} H_i}{1 - \ell_{i,i}}, \quad i = 1, 2, \dots, m.$$

PROOF. In view of Lemma 1.1 from [4], there exist the numbers $r_i > 0$, $i = 1, 2, \dots, m$ and t , $0 < t < 1$, such that

$$(9) \quad \sum_{\substack{k=1 \\ k \neq i}}^m \ell_{i,k} r_k \leq t r_i, \quad i = 1, 2, \dots, m.$$

Note that for every $\lambda > 0$ the numbers λr_i , $i = 1, 2, \dots, m$, also satisfy (9). It follows that without any loss of generality we can assume that

$$(10) \quad \text{Var}_{\langle a, b \rangle} H_i \leq (1 - t) r_i, \quad i = 1, 2, \dots, m.$$

Now let us define the system of functions spaces

$$(11) \quad X_i = \left\{ \varphi_i \in \text{BV} \langle a, b \rangle : \text{Var}_{\langle a, b \rangle} \varphi_i \leq r_i, \quad i = 1, 2, \dots, m \right\}$$

with the metrics

$$(12) \quad \varrho_i(\bar{\varphi}_i, \bar{\bar{\varphi}}_i) = \sup_{x \in \langle a, b \rangle} d(\bar{\varphi}_i(x), \bar{\bar{\varphi}}_i(x)), \quad \bar{\varphi}_i, \bar{\bar{\varphi}}_i \in X_i, \\ \text{for every } i = 1, 2, \dots, m.$$

Evidently, the space (X_i, ϱ_i) is a complete metric space, $i = 1, 2, \dots, m$.

Let us consider now the system of transformations

$$\Psi_i = T_i[\varphi_1, \dots, \varphi_m], \quad i = 1, 2, \dots, m,$$

where $\Psi_i(x)$ is defined by the formula

$$\begin{aligned}\Psi_i(x) &= h_i(x, \varphi_1[f_{i,1}(x)], \dots, \varphi_m[f_{i,m}(x)]), \\ x &\in \langle a, b \rangle, \quad i = 1, 2, \dots, m.\end{aligned}$$

We shall show that $T_i : X_1 \times X_2 \times \dots \times X_m \rightarrow X_i$, $i = 1, 2, \dots, m$.

Let $\varphi_i \in X_i$. We shall show that $\text{Var}_{\langle a, b \rangle} \Psi_i \leq r_i$, $i = 1, 2, \dots, m$. For this purpose, let us take into account the set of all partitions $P\langle a, b \rangle$ of interval $\langle a, b \rangle$ and estimate $\text{Var}_{\langle a, b \rangle} \Psi_i$.

In virtue of hypotheses (i)–(iv) and inequalities (9) and (10), we have

$$\begin{aligned}\text{Var}_{\langle a, b \rangle} \Psi_i &= \sup_{P\langle a, b \rangle} \sum_{j=1}^s d(\Psi_i(x_j), \Psi_i(x_{j-1})) \\ &= \sup_{P\langle a, b \rangle} \sum_{j=1}^s d\left(h_i(x_j, \varphi_1[f_{i,1}(x_j)], \dots, \varphi_m[f_{i,m}(x_j)]), \right. \\ &\quad \left. h_i(x_{j-1}, \varphi_1[f_{i,1}(x_{j-1})], \dots, \varphi_m[f_{i,m}(x_{j-1})])\right) \\ &\leq \sup_{P\langle a, b \rangle} \sum_{j=1}^s \left\{ \sum_{k=1}^m \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \right. \\ &\quad \left. + d(H_i(x_j), H_i(x_{j-1})) \right\} \\ &\leq \sup_{P\langle a, b \rangle} \sum_{j=1}^s \sum_{k=1}^m \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\ &\quad + \sup_{P\langle a, b \rangle} \sum_{j=1}^s d(H_i(x_j), H_i(x_{j-1})) \\ &\leq \sum_{k=1}^m \left\{ \ell_{i,k} \sup_{P\langle a, b \rangle} \sum_{j=1}^s d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \right\} \\ &\quad + \sup_{P\langle a, b \rangle} \sum_{j=1}^s d(H_i(x_j), H_i(x_{j-1})) \\ &= \sum_{k=1}^m \ell_{i,k} \text{Var}_{\langle f_{i,k}(a), f_{i,k}(b) \rangle} \varphi_k + \text{Var}_{\langle a, b \rangle} H_i\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=1}^m \ell_{i,k} \operatorname{Var}_{\langle a,b \rangle} \varphi_k + \operatorname{Var}_{\langle a,b \rangle} H_i \leq \sum_{k=1}^m \ell_{i,k} r_k + (1-t)r_i \\
&\leq tr_i + (1-t)r_i = r_i.
\end{aligned}$$

Next, we shall prove that the transformation Ψ_i is a contraction map, $i = 1, 2, \dots, m$.

Let us take an arbitrary $\bar{\varphi}_i, \bar{\bar{\varphi}}_i \in X_i$ and, taking into account (ii), (5), (9) and (12), estimate

$$\begin{aligned}
&\varrho_i(T_i[\bar{\varphi}_1, \dots, \bar{\varphi}_m], T_i[\bar{\bar{\varphi}}_1, \dots, \bar{\bar{\varphi}}_m]) = \varrho_i(\bar{\Psi}_i, \bar{\bar{\Psi}}_i) \\
&= \sup_{x \in \langle a,b \rangle} d(\bar{\Psi}_i(x), \bar{\bar{\Psi}}_i(x)) \\
&= \sup_{x \in \langle a,b \rangle} d\left(h_i(x, \bar{\varphi}_1[f_{i,1}(x)], \dots, \bar{\varphi}_m[f_{i,m}(x)]), \right. \\
&\quad \left. h_i(x, \bar{\bar{\varphi}}_1[f_{i,1}(x)], \dots, \bar{\bar{\varphi}}_m[f_{i,m}(x)])\right) \\
&\leq \sup_{x \in \langle a,b \rangle} \sum_{k=1}^m \ell_{i,k} d(\bar{\varphi}_k[f_{i,k}(x)], \bar{\bar{\varphi}}_k[f_{i,k}(x)]) \\
&\leq \sup_{x \in \langle a,b \rangle} \sum_{k=1}^m \ell_{i,k} d(\bar{\varphi}_k(x), \bar{\bar{\varphi}}_k(x)) \\
&\leq \sum_{k=1}^m \ell_{i,k} \sup_{x \in \langle a,b \rangle} d(\bar{\varphi}_k(x), \bar{\bar{\varphi}}_k(x)) = \sum_{k=1}^m \ell_{i,k} \varrho_k(\bar{\varphi}_k, \bar{\bar{\varphi}}_k).
\end{aligned}$$

Thus, the first statement of the theorem results from Theorem 1.4 in [4].

To prove that the estimation (8) holds let us take the set $P\langle a, b \rangle$ of all partitions of the interval $\langle a, b \rangle$. Using succesively (1), (iv) and (ii) we obtain

$$\begin{aligned}
\operatorname{Var}_{\langle a,b \rangle} \varphi_i &= \sup_{P\langle a,b \rangle} \sum_{j=1}^s d(\varphi_i(x_j), \varphi_i(x_{j-1})) \\
&= \sup_{P\langle a,b \rangle} \sum_{j=1}^s d\left(h_i(x_j, \varphi_1[f_{i,1}(x_j)], \dots, \varphi_m[f_{i,m}(x_j)]), \right. \\
&\quad \left. h_i(x_{j-1}, \varphi_1[f_{i,1}(x_{j-1})], \dots, \varphi_m[f_{i,m}(x_{j-1})])\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{P\langle a,b \rangle} \sum_{j=1}^s \left\{ \sum_{k=1}^m \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \right. \\
&\quad \left. + d(H_i(x_j), H_i(x_{j-1})) \right\} \\
&\leq \sup_{P\langle a,b \rangle} \sum_{j=1}^s \sum_{k=1}^m \ell_{i,k} d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \\
&\quad + \sup_{P\langle a,b \rangle} \sum_{j=1}^s d(H_i(x_j), H_i(x_{j-1})) \\
&\leq \sum_{k=1}^m \left\{ \ell_{i,k} \sup_{P\langle a,b \rangle} \sum_{j=1}^s d(\varphi_k[f_{i,k}(x_j)], \varphi_k[f_{i,k}(x_{j-1})]) \right\} \\
&\quad + \sup_{P\langle a,b \rangle} \sum_{j=1}^s d(H_i(x_j), H_i(x_{j-1})) \\
&= \sum_{k=1}^m \ell_{i,k} \operatorname{Var}_{\langle f_{i,k}(a), f_{i,k}(b) \rangle} \varphi_k + \operatorname{Var}_{\langle a,b \rangle} H_i \\
&\leq \sum_{k=1}^m \ell_{i,k} \operatorname{Var}_{\langle a,b \rangle} \varphi_k + \operatorname{Var}_{\langle a,b \rangle} H_i .
\end{aligned}$$

This completes the proof of the Theorem 1.

Remark. If in the hypothesis (iv), instead conditions (7), we assume that the characteristic roots of the matrix $[\ell_{i,k}]$ have absolute values less than 1 then Theorem 1 remain true. It follows from the Lemma 1.2 and Theorem 1.5 in [4].

2. Now let us consider the system (2) and assume the following hypotheses:

- (i) (X, d) is a complete metric space,
- (ii) $f : (a, b) \rightarrow (a, b)$ is continuous, strictly increasing in (a, b) and $a < f(x) < x$ for $x \in (a, b)$,
- (iii) $g_i : (a, b) \times X^m \rightarrow X$, $i = 1, 2, \dots, m$,

- (iv) There are functions $G_i \in \text{BV}(a, b)$ and positive constants $\ell_{i,k}$, $i, k = 1, 2, \dots, m$, such that

$$\begin{aligned} & d(g_i(x, y_1, \dots, y_m), g_i(\bar{x}, \bar{y}_1, \dots, \bar{y}_m)) \\ & \leq \sum_{k=1}^m \ell_{i,k} d(y_k, \bar{y}_k) + d(G_i(x), G_i(\bar{x})), \end{aligned}$$

$$i = 1, 2, \dots, m, (x, y_1, \dots, y_m), (\bar{x}, \bar{y}_1, \dots, \bar{y}_m) \in (a, b) \times X^m,$$

- (v) All the characteristic roots of the matrix $[\ell_{i,k}]$ have absolute values less than 1.

We shall prove the following:

Theorem 2. *Let hypotheses (i)–(v) be fulfilled. Then the system of equations (2) has in the interval (a, b) a solution $\varphi = (\varphi_1, \dots, \varphi_m)$, $\varphi_i \in \text{BV}(a, b)$, $i = 1, 2, \dots, m$, depending on an arbitrary function. More precisely: for any system of functions $\{\varphi_{i,0}\} : \langle f(b), b \rangle \rightarrow \mathbb{R}$, such that $\varphi_{i,0} \in \text{BV}\langle f(b), b \rangle$, $i = 1, 2, \dots, m$, there exists the unique system of functions $\varphi = (\varphi_1, \dots, \varphi_m)$, $\varphi_i \in \text{BV}(a, b)$ satisfying the system of equations (2) in (a, b) and such that $\varphi_i = \varphi_{i,0}$ in the interval $\langle f(b), b \rangle$, $i = 1, 2, \dots, m$.*

PROOF. Let $I_n = \langle f^{n+1}(b), f^n(b) \rangle$, $n = 0, 1, 2, \dots$, where $f^n(x)$ denotes the n -th iterate of the function $f(x)$. In virtue of hypothesis (ii) we have $\bigcup_{n=0}^{\infty} I_n = (a, b)$. In the interval $I_0 = \langle f(b), b \rangle$ we define an arbitrary system of functions $\{\varphi_{i,0}\}$, $i = 1, 2, \dots, m$, fulfilling the conditions of the theorem. Notice, that if $x \in I_1$ then $f^{-1}(x) \in I_0$. So let us define the function $\varphi_{i,1} : I_1 \rightarrow \mathbb{R}$ in the following way:

$$\begin{aligned} \varphi_{i,1}(x) &= g_i(f^{-1}(x), \varphi_{1,0}[f^{-1}(x)], \dots, \varphi_{m,0}[f^{-1}(x)]), \\ x &\in I_1, \quad i = 1, 2, \dots, m. \end{aligned}$$

We shall show that $\varphi_{i,1} \in \text{BV}(I_1)$. Let $P(I_1)$ be the set of all finite partitions of the interval I_1 . From the hypotheses (iv) and (ii) we get successively

$$\begin{aligned} \text{Var}_{I_1} \varphi_{i,1} &= \sup_{P(I_1)} \sum_{j=1}^s d(\varphi_{i,1}(x_j), \varphi_{i,1}(x_{j-1})) \\ &= \sup_{P(I_1)} \sum_{j=1}^s d(g_i(f^{-1}(x_j), \varphi_{1,0}[f^{-1}(x_j)], \dots, \varphi_{m,0}[f^{-1}(x_j)]), \\ &\quad g_i(f^{-1}(x_{j-1}), \varphi_{1,0}[f^{-1}(x_{j-1})], \dots, \varphi_{m,0}[f^{-1}(x_{j-1})])) \end{aligned}$$

$$\begin{aligned}
&\leq \sup_{P(I_1)} \sum_{j=1}^s \left\{ \sum_{k=1}^m \ell_{i,k} d(\varphi_{k,0}[f^{-1}(x_j)], \varphi_{k,0}[f^{-1}(x_{j-1})]) \right. \\
&\quad \left. + d(G_i[f^{-1}(x_j)], G_i[f^{-1}(x_{j-1})]) \right\} \\
&\leq \sup_{P(I_1)} \sum_{j=1}^s \sum_{k=1}^m \ell_{i,k} d(\varphi_{k,0}[f^{-1}(x_j)], \varphi_{k,0}[f^{-1}(x_{j-1})]) \\
&\quad + \sup_{P(I_1)} \sum_{j=1}^s d(G_i[f^{-1}(x_j)], G_i[f^{-1}(x_{j-1})]) \\
&\leq \sum_{k=1}^m \left\{ \ell_{i,k} \sup_{P(I_1)} \sum_{j=1}^s d(\varphi_{k,0}[f^{-1}(x_j)], \varphi_{k,0}[f^{-1}(x_{j-1})]) \right\} \\
&\quad + \sup_{P(I_1)} \sum_{j=1}^s d(G_i[f^{-1}(x_j)], G_i[f^{-1}(x_{j-1})]) \\
&= \sum_{k=1}^m \ell_{i,k} \underset{\langle f(b), b \rangle}{\text{Var}} \varphi_{k,0} + \underset{\langle f(b), b \rangle}{\text{Var}} G_i = \sum_{k=1}^m \ell_{i,k} \underset{I_0}{\text{Var}} \varphi_{k,0} + \underset{I_0}{\text{Var}} G_i.
\end{aligned}$$

Hence $\varphi_{i,1} \in \text{BV}(I_1)$.

More generally, let us take the interval I_{n+1} . If $x \in I_{n+1}$ then $f^{-1}(x) \in I_n$. Let us define a function $\varphi_{i,n+1} : I_{n+1} \rightarrow \mathbb{R}$ in the following way:

$$\begin{aligned}
(13) \quad \varphi_{i,n+1}(x) &= g_i(f^{-1}(x), \varphi_{1,n}[f^{-1}(x)], \dots, \varphi_{m,n}[f^{-1}(x)]), \\
&\quad x \in I_{n+1}, \quad i = 1, 2, \dots, m.
\end{aligned}$$

By similar calculation as above one can prove that $\varphi_{i,n+1} \in \text{BV}(I_{n+1})$ and that

$$(14) \quad \underset{I_{n+1}}{\text{Var}} \varphi_{i,n+1} \leq \sum_{k=1}^m \ell_{i,k} \underset{I_n}{\text{Var}} \varphi_{k,n} + \underset{I_n}{\text{Var}} G_i, \quad i = 1, 2, \dots, m.$$

Now we introduce the following notations:

$$(15) \quad a_{i,k} = \underset{I_k}{\text{Var}} \varphi_{i,k} \quad b_{i,k} = \underset{I_k}{\text{Var}} G_i$$

Using these notations we can write the inequality (14) as follows:

$$(16) \quad a_{i,n+1} \leq \sum_{k=1}^m \ell_{i,k} a_{k,n} + b_{i,n}, \quad i = 1, 2, \dots, m.$$

In the interval (a, b) we define the function $\varphi = (\varphi_1, \dots, \varphi_m)$ in the following way:

$$(17) \quad \varphi_i(x) = \begin{cases} \varphi_{i,0}(x) & \text{for } x \in I_0, \quad i = 1, 2, \dots, m \\ \varphi_{i,n+1}(x) & \text{for } x \in I_{n+1}, \quad i = 1, 2, \dots, m, \\ & n = 0, 1, \dots \end{cases}$$

From (13) and (17) it follows that φ satisfies the system of equations (2). The uniqueness of the solution is obvious.

We have to prove that $\varphi_i \in \text{BV}(a, b)$, $i = 1, 2, \dots, m$. From (15) and (17) it follows that

$$\begin{aligned} \text{Var}_{(a,b)} \varphi_i &= \sum_{n=0}^{\infty} \text{Var}_{I_n} \varphi_i = \text{Var}_{I_0} \varphi_i + \sum_{n=1}^{\infty} \text{Var}_{I_n} \varphi_i \\ &= \text{Var}_{I_0} \varphi_{i,0} + \sum_{n=1}^{\infty} \text{Var}_{I_n} \varphi_{i,n} = \text{Var}_{I_0} \varphi_{i,0} + \sum_{n=i}^{\infty} a_{i,n}. \end{aligned}$$

Since, by hypotheses of the theorem, $\text{Var}_{I_0} \varphi_{i,0}$ is finite, and every component of the series $\sum_{n=1}^{\infty} a_{i,n}$ fulfils inequality (16), in view of Lemma 4.1 from the paper [4], this series converges. This completes the proof of Theorem 2.

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