

On the zeros of solutions of ordinary second order differential equations

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We consider the solution of the differential equation

$$(1) \quad y'' + q(x)y = 0,$$

where $q(x)$, $q'(x)$, $q''(x)$ are continuous functions, $q(x) > 0$, $q'(x) > 0$ for $x > 0$, and

$$(2) \quad \alpha q'^2(x) - q(x)q''(x) \geq 0 \quad \text{for } x > 0,$$

with the initial conditions $y(0) = 0$, $y'(0) \neq 0$. Using the notation $\sigma = \sigma(x) = \sqrt{q(x)}$ we obtain from (2)

$$(3) \quad (2\alpha - 1)\sigma'^2(x) - \sigma(x)\sigma''(x) \geq 0 \quad \text{for } x > 0,$$

where we put in the following $\beta = 2\alpha - 1$.

We denote the first n roots of the equation $y(x) = 0$, if they exist, by $x_0 = 0, x_1, \dots, x_n$, those of $y'(x) = 0$ by $x'_1, x'_2, \dots, x'_n$, where

$$x_0 = 0 < x'_1 < x_1 < \dots < x'_n < x_n.$$

For the root x_i it follows from a result of E. MAKAI (see [1]), that the inequality

$$(4) \quad \int_0^{x_i} \sigma(x) dx < i\pi \quad (i = 1, 2, \dots, n)$$

holds, if α is not greater than $\frac{5}{4}$.

Makai conjectured (see [2]) that the quantities on the left of (4) are in certain circumstances greater than $(i - \frac{1}{2})\pi$. In this paper we will prove the following

Theorem. *If the inequality $\alpha < 1$ holds for the quantity α in (2) then the inequalities*

$$(5) \quad \begin{cases} \int_0^{x_i} \sigma(x) dx > i\pi - \frac{1}{3-2\alpha} \frac{\pi}{2} & (i = 1, 2, \dots, n), \\ \int_0^{x'_i} \sigma(x) dx > \left(i - \frac{1}{2}\right)\pi - \frac{1}{3-2\alpha} \frac{\pi}{2} & (i = 1, 2, \dots, n) \end{cases}$$

hold, and if $1 \leq \alpha \leq 3/2$ then

$$(6) \quad \begin{cases} \int_0^{x_i} \sigma(x) dx > i\pi - \frac{\pi}{2} & (i = 1, 2, \dots, n), \\ \int_0^{x'_i} \sigma(x) dx > \left(i - \frac{1}{2}\right)\pi - \frac{\pi}{2} & (i = 1, 2, \dots, n). \end{cases}$$

PROOF. Let us introduce the continuous function

$$(7) \quad \varphi(x) = \operatorname{arctg} \frac{\sigma(x)y(x)}{y'(x)}$$

and let be

$$(8) \quad \varphi(0) = 0.$$

It is easy to see that this function satisfies the differential equation

$$(9) \quad \varphi'(x) = \sigma(x) + \frac{\sigma'(x)}{2\sigma(x)} \sin 2\varphi(x).$$

We will prove that the inequality

$$(10) \quad \varphi'(x) > 0, \quad 0 < x < x_n$$

holds. It follows simply from the definition of $\varphi(x)$ that the inequality $0 < \varphi(x) \leq \pi/2$ is true provided $0 < x \leq x'_1$ hence (10) is true for $0 < x \leq x'_1$.

Assuming that (10) does not hold always we define a by

$$a = \min \{x; x'_1 \leq x \leq x_n, \varphi'(x) \leq 0\}.$$

In this case we have

$$(11) \quad \varphi'(x) > 0$$

for every $0 < x < a$. Since

$$(12) \quad \begin{cases} \varphi'(a) = \sigma(a) + \frac{\sigma'(a)}{2\sigma(a)} \sin 2\varphi(a) = 0, \\ \varphi''(x) = \sigma'(x) + \left[\frac{\sigma'(x)}{\sigma(x)} \right]' \frac{1}{2} \sin 2\varphi(x) + \frac{\sigma'(x)}{\sigma(x)} \cos 2\varphi(x) \cdot \varphi'(x), \end{cases}$$

we obtain, taking into account (3), that

$$\varphi''(a) = \frac{2\sigma'^2(a) - \sigma(a)\sigma''(a)}{\sigma'(a)} > \frac{\beta\sigma'^2(a) - \sigma(a)\sigma''(a)}{\sigma'(a)} \equiv 0.$$

From the continuity of $\varphi''(x)$ it follows that if $a - \varepsilon < x < a$ and ε is sufficiently small, then $\varphi''(x) > 0$ and

$$0 < \int_x^a \varphi''(x) dx = -\varphi'(x).$$

Consequently $\varphi'(x) < 0$, which contradicts to (11), therefore (10) is true.

On the basis of (7), (8) and (10) we have

$$(13) \quad \begin{cases} \varphi(x_i) = i\pi & i = 0, 1, 2, \dots, n \\ \varphi(x'_i) = \left(i - \frac{1}{2}\right)\pi & i = 1, 2, \dots, n. \end{cases}$$

Integrating the differential equation (9) between 0 and x we obtain

$$(14) \quad \varphi(x) = J(x) + F(x)$$

where

$$J(x) = \int_0^x \sigma(u) du \quad \text{and} \quad F(x) = \int_0^x \frac{\sigma'(u)}{2\sigma(u)} \sin 2\varphi(u) du.$$

It can be easily seen that $F(x)$ takes on its local maxima at $x'_1, x'_2, \dots, x'_n$ and its local minima at $x_0=0, x_1, \dots, x_n$. We will prove that

$$(15) \quad F(x'_i) > F(x'_{i+1}) \quad i = 1, 2, \dots, n-1.$$

Now we have taking (9) into account

$$\begin{aligned} F(x'_{i+1}) - F(x'_i) &= \int_{x'_i}^{x'_{i+1}} \frac{\sigma'(x)}{2\sigma(x)} \sin 2\varphi(x) dx = \\ &= \int_{x'_i}^{x'_{i+1}} \frac{\sin 2\varphi(x)}{\frac{2\sigma(x)}{\sigma'(x)} \cdot \varphi'(x)} \varphi'(x) dx = \int_{x'_i}^{x'_{i+1}} \frac{\sin 2\varphi(x)}{\frac{2\sigma^2(x)}{\sigma'(x)} + \sin 2\varphi(x)} \varphi'(x) dx. \end{aligned}$$

The function $s(\varphi(x)) \stackrel{\text{def}}{=} 2\sigma^2(x)/\sigma'(x)$ is steadily increasing because from our restriction for α and from (3)

$$\left[\frac{\sigma^2(x)}{\sigma'(x)} \right]' = \frac{\sigma(x)}{\sigma'^2(x)} [2\sigma'^2(x) - \sigma(x)\sigma''(x)] > 0.$$

So we have

$$\begin{aligned} F(x'_{i+1}) - F(x'_i) &= \int_{(i-\frac{1}{2})\pi}^{(i+\frac{1}{2})\pi} \frac{\sin 2\varphi}{s(\varphi) + \sin 2\varphi} d\varphi = \\ &= \int_{i\pi}^{(i+\frac{1}{2})\pi} \left[\frac{\sin 2\varphi}{s(\varphi) + \sin 2\varphi} - \frac{\sin 2\varphi}{s(\varphi - \frac{\pi}{2}) - \sin 2\varphi} \right] d\varphi = \\ &= - \int_{i\pi}^{(i+\frac{1}{2})\pi} \sin 2\varphi \frac{\left[s(\varphi) - s(\varphi - \frac{\pi}{2}) \right] + 2 \sin 2\varphi}{[s(\varphi) + \sin 2\varphi] [s(\varphi - \frac{\pi}{2}) - \sin 2\varphi]} d\varphi < 0 \end{aligned}$$

hence the inequalities (15) and

$$(16) \quad F(x) \leq F(x'_1), \quad x \geq 0$$

hold. ¹⁾ We obtain another inequality for $F(x'_1)$ from (14) by putting $x = x'_1$:

$$\frac{\pi}{2} = J(x'_1) + F(x'_1) > F(x'_1)$$

and $J(x) = \varphi(x) - F(x) \geq \varphi(x) - F(x'_1) > \varphi(x) - \frac{\pi}{2}$. The inequalities (6) follow from this last inequality by putting in it $x = x_i$ and $x = x'_i$ ($i = 1, 2, \dots, n$) respectively. To prove the statements (5) a more precise estimation is needed.

Now we want to prove the validity of the inequality

$$(17) \quad F(x'_1) < \frac{1}{3-2\alpha} \frac{\pi}{2}$$

if $\alpha < 1$.

This is sufficient for our purpose because from (14) and (16) we have

$$J(x_i) = \varphi(x_i) - F(x_i) > \varphi(x_i) - F(x'_1), \quad i = 1, 2, \dots, n$$

and

$$J(x'_i) = \varphi(x'_i) - F(x'_i) \geq \varphi(x'_i) - F(x'_1), \quad i = 1, 2, \dots, n,$$

and taking into account (13) and (17) we obtain (5). At first we need the inequality

$$(18) \quad \varphi(x) < x\sigma(x) \quad \text{for } x > 0.$$

From (7) we have $\sigma(x) = \operatorname{tg} \varphi(x) \cdot y'(x)/y(x)$ therefore

$$(19) \quad \lim_{x \rightarrow 0} \frac{\varphi(x)}{\sigma(x)} = \lim_{x \rightarrow 0} \frac{\varphi(x)}{\operatorname{tg} \varphi(x)} \cdot \frac{y(x)}{y'(x)} = 1 \cdot \frac{0}{y'(0)} = 0.$$

Again, it follows from equation (9), by using the well known inequality $\sin x < x$ for $x > 0$ that

$$\varphi'(x) < \sigma(x) + \frac{\sigma'(x)\varphi(x)}{\sigma(x)}$$

or $[\varphi(x)/\sigma(x)]' < 1$ ($x > 0$). Integrating this between 0 and x and taking into account (19) we obtain $\varphi(x)/\sigma(x) < x$, i.e. the inequality (18).

We now derive an inequality, which we shall need in proving the inequality (17). If $\sigma(x) > 0$, $\beta < 1$, $\beta\sigma'^2(x) - \sigma(x)\sigma''(x) \geq 0$ for $a < x < b$ then

$$(20) \quad \int_a^b \sigma(x) dx \geq \sigma(b)(b-a) \frac{1-\beta}{2-\beta}.$$

¹⁾ In the case $\alpha < 5/4$ there exist inequalities among $F(x_0)$, $F(x_1)$, ..., $F(x_n)$ similar to (15). E. Makai proved (see [1]) that with our notations

$$F(x_{i+1}) - F(x_i) = \varphi(x_{i+1}) - \varphi(x_i) - [Y(x_{i+1}) - Y(x_i)] = \pi - \int_{x_i}^{x_{i+1}} \sigma(x) dx > 0,$$

hence

$$0 = F(x_0) < F(x_1) < \dots < F(x_n).$$

Indeed, let us consider the function $t(x) \stackrel{\text{def}}{=} [\sigma(x)]^{1-\beta}$. This function is concave because its second derivative $t''(x) = (1-\beta)\sigma^{-1-\beta}[\sigma\sigma'' - \beta\sigma'^2]$ is not more than 0 by our assumptions, so that

$$t(x) \cong \frac{t(b)}{b-a}(x-a) \quad \text{for } a \leq x \leq b,$$

hence

$$\sigma(x) \cong \frac{\sigma(b)}{(b-a)^{\frac{1}{1-\beta}}}(x-a)^{\frac{1}{1-\beta}} \quad \text{for } a \leq x \leq b.$$

Integrating this over $[a, b]$, we obtain

$$\int_a^b \sigma(x) dx \cong \int_a^b \frac{\sigma(b)}{(b-a)^{\frac{1}{1-\beta}}}(x-a)^{\frac{1}{1-\beta}} dx = \frac{1-\beta}{2-\beta} \sigma(b)(b-a),$$

i.e. inequality (20). Since in our case $\beta\sigma'^2(x) - \sigma(x)\sigma''(x) \cong 0$, where $\beta = 2\alpha - 1 < 1$, $\sigma(x) > 0$ for $0 < x < x_n$, we have from (20) by applying (18) at $x = x_1$

$$J(x'_1) = \int_0^{x'_1} \sigma(x) dx \cong \frac{1-\beta}{2-\beta} x'_1 \sigma(x'_1) > \frac{1-\beta}{2-\beta} \frac{\pi}{2}.$$

It follows from (14) and from the last inequality that

$$(21) \quad F(x'_1) = \frac{\pi}{2} - J(x'_1) < \frac{1}{2-\beta} \frac{\pi}{2},$$

which is the same as (17), since $\beta = 2\alpha - 1$.

We mention the interesting case $q''(x) \leq 0$ when $\alpha = 0$. In this case $F(x'_1) < \pi/6$.

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References

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