

On Taylor series absolutely convergent on the circumference of the circle of convergence II.

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Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

be regular for $|z| < 1$ and substitute z by a conformal mapping of $|w| < 1$ onto $|z| < 1$ different from a rotation:

$$z = T(w) = e^{i\theta} \frac{w - \zeta}{1 - \bar{w}\bar{\zeta}} \quad (\theta \text{ real, } 0 < |\zeta| < 1),$$

$$f[T(w)] = \sum_{n=0}^{\infty} b_n w^n.$$

L. ALPÁR [1] proved that the absolute convergence of $\sum_{k=0}^{\infty} a_k z^k$ on the periphery is not necessarily inherited by $\sum_{n=0}^{\infty} b_n w^n$. Generalizing his result, in part I. *) we constructed examples with $\sum_{k=0}^{\infty} |a_k| < +\infty$ where $\sum_{n=0}^{\infty} |b_n|$ not simply diverged but did so in a certain sense rapidly. In this paper we are concerned with extra conditions imposed on the original series $\sum_{k=0}^{\infty} a_k z^k$.

Theorem 1. If $\sum_{k=0}^{\infty} |a_k|$ has a convergent majorant series of decreasing terms:

$$|a_k| \leq A_k, \quad A_{k+1} \leq A_k, \quad \sum_{k=0}^{\infty} A_k < +\infty$$

or what is the same thing

$$\sum_{k=0}^{\infty} \max_{l \geq k} |a_l| < +\infty,$$

then always

$$\sum_{n=0}^{\infty} |b_n| < +\infty.$$

*) *Publ. Math. (Debrecen)* **14** (1967), 63—68.

If the majorant is only a little greater we have, on the contrary,

Theorem 2. If a series $\sum_{k=0}^{\infty} A_k$ is given with

$$A_{k+1} \leq A_k, \quad \sum_{k=0}^{\infty} A_k = +\infty$$

then a_k can be chosen to satisfy the conditions ¹⁾

$$|a_k| \leq A_k, \quad \sum_{k=0}^{\infty} |a_k| < +\infty, \quad \sum_{n=0}^{\infty} |b_n| = +\infty.$$

In that follows $c, c_1, c_2, \dots$ denote positive constants which may possibly depend on quantities we think to be fixed (such as ζ in the expression of $T(w)$ e.t.c.) The notation $O(\dots)$ will be understood similarly.

PROOF OF THEOREM 1. Putting the Taylor series of $T(w)$ into that of $f(z)$, we find

$$b_n = \sum_{k=0}^{\infty} t_{kn} a_k,$$

t_{kn} being the n th coefficient of the k th power of $T(w)$.

Let $\zeta = re^{i\varphi}$ and fix somehow p and q subject only to the conditions $0 < p < \frac{1-r}{1+r} < \frac{1+r}{1-r} < q$, c.g. by putting $p = \frac{1}{2} \frac{1-r}{1+r}$, $q = 2 \frac{1+r}{1-r}$. As a first step we prove

$$(1) \quad \sum_{k \leq pn} + \sum_{k \geq qn} |t_{kn}| \leq c_1 e^{-c_2 n}.$$

We may confine ourselves to the first sum, the treatment of the second being similar.

According to the definition of t_{kn} , by Cauchy's formula

$$t_{kn} = \frac{1}{2\pi i} \int_{|w|=1} \frac{T^k(w)}{w^{n+1}} dw.$$

Let $\delta > 0$, to be determined later and $\eta > 0$ sufficiently small depending on δ and let us deform the path of integration into the circle of radius $1 + \eta$ ²⁾. This is possible if the pole of $T(w)$ lies outside, that is $1 + \eta < \frac{1}{|\zeta|} = \frac{1}{r}$. A simple analysis of the

¹⁾ Theorem 2 disproves the following conjecture of Mr. SHIELDS: „ $\sum |a_k| < +\infty$ and $\sum k|a_k|^2 < +\infty$ imply $\sum |b_n| < +\infty$ ” since we can put $A_k = \frac{1}{k \log k}$ ($k \geq 2$) and then $\sum_2^{\infty} k|a_k|^2 \leq \sum_2^{\infty} k \frac{1}{k^2 \log^2 k} < +\infty$. As we were informed a better example in this direction had been constructed by Messrs. KAHANE and KATZNELSON.

²⁾ For the range $k \geq qn$ we would choose $1 - \eta$.

behaviour of the mapping $T(w)$ tells us that on this circle it takes the largest absolute value at $(1+\eta)e^{i\varphi}$ ($\varphi = \arg \zeta$) namely

$$|T([1+\eta]e^{i\varphi})| = \left| \frac{(1+\eta)e^{i\varphi} - re^{i\varphi}}{1 - (1+\eta)e^{i\varphi}re^{-i\varphi}} \right| = \frac{1-r+\eta}{1-r-\eta r} = 1 + \eta \frac{1+r}{1-r-\eta r} \leq e^{\eta \left(\frac{1+r}{1-r} + \delta \right)}$$

while for the constant $|w|$

$$|w| = 1 + \eta \leq e^{\eta(1-\delta)}$$

so that we have

$$\begin{aligned} |t_{kn}| &\leq \frac{1}{2\pi} \int_{|w|=1+\eta} \frac{|T(w)|^k}{|w|^{n+1}} |dw| \leq e^{k\eta \left(\frac{1+r}{1-r} + \delta \right)} e^{-n\eta(1-\delta)}, \\ \sum_{k \leq pn} |t_{kn}| &\leq e^{-n\eta(1-\delta)} \sum_{k \leq pn} e^{k\eta \left(\frac{1+r}{1-r} + \delta \right)} \leq e^{-n\eta(1-\delta)} \frac{e^{pn\eta \left(\frac{1+r}{1-r} + \delta \right)}}{1 - e^{-\eta \left(\frac{1+r}{1-r} + \delta \right)}} = \\ &= \frac{1}{1 - e^{-\eta \left(\frac{1+r}{1-r} + \delta \right)}} e^{-n\eta \left(1 - p \frac{1+r}{1-r} - [1+p]\delta \right)} = c_1 e^{-c_2 n} \end{aligned}$$

if we first fix $\delta > 0$ with $1 - p \frac{1+r}{1-r} - [1+p]\delta > 0$ (take notice of $p < \frac{1-r}{1+r}$) and then $\eta > 0$ in accordance with δ and $1 + \eta < \frac{1}{r}$.

Now we deal with b_n on the range $N < n \leq 2N$. Let us put

$$a_k^* = a_k \quad \text{if } pN < k < 2qN$$

$$a_k^* = 0 \quad \text{otherwise,}$$

$$f^*(z) = \sum_{k=0}^{\infty} a_k^* z^k = \sum_{pN < k < 2qN} a_k z^k,$$

$$f^*(T[w]) = \sum_{n=0}^{\infty} b_n^* w^n.$$

We have for $N < n \leq 2N$ (since $|a_k| \leq c_3$)

$$\begin{aligned} |b_n - b_n^*| &= \left| \sum_{k=0}^{\infty} t_{kn}(a_k - a_k^*) \right| = \left| \sum_{k \leq pN} + \sum_{k \geq 2qN} t_{kn} a_k \right| \leq c_3 \left\{ \sum_{k \leq pN} + \sum_{k \geq 2qN} |t_{kn}| \right\} \leq \\ &\leq c_3 \left\{ \sum_{k \leq pn} + \sum_{k \geq qn} |t_{kn}| \right\} \leq c_4 e^{-c_2 n} \leq c_4 e^{-c_2 N} \end{aligned}$$

by what we have just proved. This estimation suggests that b_n can be replaced by b_n^* .

For this latter by Parseval's formula

$$\begin{aligned} \sum_{n=0}^{\infty} |b_n^*|^2 &= \frac{1}{2\pi} \int_{|w|=1} |f^*(T[w])|^2 |dw| = \frac{1}{2\pi} \int_{|z|=1} |f^*(z)|^2 \left| \frac{dw}{dz} \right| |dz| \leq \\ &\leq \max_{|z|=1} \left| \frac{dw}{dz} \right| \cdot \frac{1}{2\pi} \int_{|z|=1} |f^*(z)|^2 |dz| = c_s \sum_{k=0}^{\infty} |a_k^*|^2 = c_s \sum_{pN < k < 2qN} |a_k|^2. \end{aligned}$$

Using the majorant of a_k and Schwartz's inequality, we get further

$$\begin{aligned} \sum_{N < n \leq 2N} |b_n^*| &\leq \sqrt{N \sum_{n=0}^{\infty} |b_n^*|^2} \leq \sqrt{N c_s \sum_{pN < k < 2qN} |a_k|^2} \leq \sqrt{N c_s 2N q A_{[pN]}^2} = c_6 N A_{[pN]} \leq \\ &\leq c_7 \sum_{\frac{1}{2}pN < k \leq pN} A_k \quad \left(N \geq \frac{1}{p} \right). \end{aligned}$$

Finally, using this and the bound for $|b_n - b_n^*|$

$$\sum_{N < n \leq 2N} |b_n| \leq c_8 \{ N e^{-c_2 N} + \sum_{\frac{1}{2}pN < k \leq pN} A_k \}$$

and summing for $N = 1, 2, 4, 8, \dots$

$$\sum_{n=2}^{\infty} |b_n| \leq c_8 \left\{ \sum_{l=0}^{\infty} 2^l e^{-c_2 2^l} + \left| \sum_{k=1}^{\infty} A_k \right| \right\} < +\infty,$$

q.e.d.

Remark. From the proof we see, choosing $A_k = \max_{l \geq k} |a_l|$ that

$$\sum_{n=0}^{\infty} |b_n| \leq c \sum_{k=0}^{\infty} \max_{l \geq k} |a_l|$$

where c is depending only on the transformation $T(w)$.

PROOF OF THEOREM 2. Let us regard as a test function

$$g_m(z) = \frac{1}{1 - (\varrho z)^m} = \sum_{l=0}^{\infty} (\varrho z)^{lm} \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} u_k^{(m)} z^k, \quad (|\varrho| = 1 \text{ const}, m \geq 1)$$

and put

$$(2) \quad g_m(T[w]) = \frac{1}{1 - [\varrho T(w)]^m} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} v_n^{(m)} w^n.$$

We have

$$\sum_{k=0}^N |u_k^{(m)}| = \frac{N}{m} + O(1)$$

and if we can show that with ϱ suitably chosen and for large m

$$\sum_{n=0}^N |v_n^{(m)}|$$

will be much greater than this, then we have good enough functions $g_m(z)$ for which $g_m(T[w])$ behave much worse. Once we are in possession of such functions the actual construction of an example required by Theorem 2 will be a straightforward one.

Actually we prove the following. There exists a ϱ ($|\varrho|=1$) depending only on $T(w)$ such that for every $m \geq 1$

$$(3) \quad \sum_{\alpha N < n \leq N} |v_n^{(m)}| \leq c_9 \frac{N}{\sqrt{m}} \quad \text{whenever} \quad N \geq N_0(m).$$

$0 \leq \alpha < 1$ will be fixed in the application and so c_9 is also fixed.

To begin with, let us derive this inequality from

$$(4) \quad \sum_{\alpha N < n \leq N} |v_n^{(m)}|^2 \leq c_{10} \frac{N}{m},$$

$$(5) \quad \sum_{\alpha N < n \leq N} |v_n^{(m)}|^4 \leq c_{11} \frac{N}{m^2}$$

($N \geq N_0(m)$). In fact, by Hölder's inequality

$$\begin{aligned} \sum_{\alpha N < n \leq N} |v_n^{(m)}|^{2/3} |v_n^{(m)}|^{4/3} &\leq \left(\sum_{\alpha N < n \leq N} |v_n^{(m)}|^2 \right)^{2/3} \left(\sum_{\alpha N < n \leq N} |v_n^{(m)}|^4 \right)^{1/3}, \\ \sum_{\alpha N < n \leq N} |v_n^{(m)}| &\leq \frac{\left(\sum_{\alpha N < n \leq N} |v_n^{(m)}|^2 \right)^{3/2}}{\left(\sum_{\alpha N < n \leq N} |v_n^{(m)}|^4 \right)^{1/2}} \leq \frac{\left(c_{10} \frac{N}{m} \right)^{3/2}}{\left(c_{11} \frac{N}{m^2} \right)^{1/2}} = c_9 \frac{N}{\sqrt{m}}. \end{aligned}$$

To prove (4) and (5), we represent $v_n^{(m)}$ as a finite sum. Its definition (2) and Cauchy's formula give us

$$v_n^{(m)} = \frac{1}{2\pi i} \int \frac{g_m(T[w])}{w^{n+1}} dw = \frac{1}{2\pi i} \int \frac{1}{w^{n+1} (1 - [\varrho T(w)]^m)} dw,$$

integrated on a closed curve round the origin in $|w| < 1$. Substituting $w = S(z)$ where $S(z)$ is the inverse mapping of $T(w)$:

$$(6) \quad S(z) = e^{-i\vartheta} \frac{z + \zeta e^{i\vartheta}}{1 + \zeta \bar{\zeta} e^{-i\vartheta}},$$

$$v_n^{(m)} = \frac{1}{2\pi i} \int \frac{1}{S^{n+1}(z) [1 - (\varrho z)^m]} S'(z) dz,$$

the integration is over a closed curve round $-\zeta e^{i\vartheta}$ in $|z| < 1$. Replacing this path by $|z| = R$, $R \rightarrow \infty$ we find that the integral on $|z| = R$ tends to zero so that we have only to collect residues outside the original path. $\frac{S'(z)}{S^{n+1}(z)}$ is (for $n \geq 1$) regular

there, while $\frac{1}{1-(\varrho z)^m}$ has poles of the first order at $\varrho^{-1}\eta$ (η denoting m th unit roots) with residues $-\frac{1}{\varrho m \eta^{m-1}} = -\frac{\eta}{m\varrho}$ hence

$$v_n^{(m)} = \frac{1}{m\varrho} \sum_{\eta} \frac{S'(\varrho^{-1}\eta)}{S^{n+1}(\varrho^{-1}\eta)} \eta \quad (n \geq 1).$$

Here $S(\varrho^{-1}\eta)$ are numbers of unit absolute value.

Turning to (4) we have

$$\sum_{\alpha N < n \leq N} |v_n^{(m)}|^2 = \sum_{\alpha N < n \leq N} \frac{1}{m^2} \sum_{\eta_1, \eta_2} \frac{S'(\varrho^{-1}\eta_1) \bar{S}'(\varrho^{-1}\eta_2)}{S^{n+1}(\varrho^{-1}\eta_1) \bar{S}^{n+1}(\varrho^{-1}\eta_2)} \eta_1 \bar{\eta}_2 = \frac{1}{m^2} \sum_{\eta_1, \eta_2} \sum_n.$$

The inner sum is a geometric series with quotient $\frac{1}{S(\varrho^{-1}\eta_1) \bar{S}(\varrho^{-1}\eta_2)} = \frac{S(\varrho^{-1}\eta_2)}{S(\varrho^{-1}\eta_1)}$.

This is 1 if $\eta_1 = \eta_2$ and all such terms give

$$\frac{1}{m^2} \sum_{\eta} |S'(\varrho^{-1}\eta)|^2 (N - [\alpha N]) \geq \frac{N(1-\alpha)}{m^2} \cdot m \cdot \min_{|z|=1} |S'(z)|^2 = c_{13} \frac{N}{m}.$$

Here by conformity $S'(z) \neq 0$, $c_{13} > 0$.

Again by conformity, the numbers $S(\varrho^{-1}\eta)$ are all different hence, on the other hand, the quotient is different from 1 if $\eta_1 \neq \eta_2$, though of unit absolute value and the geometric series is bounded by a quantity independent of N . Summation for such (η_1, η_2) is also independent of N , therefore the remaining terms give a quantity bounded by a constant depending only on m (once ϱ is determined). If N is so large that $c_{13} \frac{N}{m}$ is greater than the double of this bound we get (4) with $c_{10} = \frac{c_{13}}{2}$.

For (5) we have

$$\begin{aligned} & \sum_{\alpha N < n \leq N} |v_n^{(m)}|^4 = \\ &= \sum_{\alpha N < n \leq N} \frac{1}{m^4} \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \frac{S'(\varrho^{-1}\eta_1) S'(\varrho^{-1}\eta_2) \bar{S}'(\varrho^{-1}\eta_3) \bar{S}'(\varrho^{-1}\eta_4)}{[S(\varrho^{-1}\eta_1) S(\varrho^{-1}\eta_2) \bar{S}(\varrho^{-1}\eta_3) \bar{S}(\varrho^{-1}\eta_4)]^{n+1}} \eta_1 \eta_2 \bar{\eta}_3 \bar{\eta}_4 = \\ &= \frac{1}{m^4} \sum_{\eta_1, \eta_2, \eta_3, \eta_4} \sum_n. \end{aligned}$$

The inner sum is again a geometric series. Its quotient is certainly 1 if $\eta_1 = \eta_3$, $\eta_2 = \eta_4$ or $\eta_1 = \eta_4$, $\eta_2 = \eta_3$. The contribution of these terms is less than

$$2 \frac{1}{m^4} \sum_{\eta_1, \eta_2} |S'(\varrho^{-1}\eta_1)|^2 |S'(\varrho^{-1}\eta_2)|^2 (N - [\alpha N]) \leq \frac{2N}{m^4} m^2 \max_{|z|=1} |S'(z)|^4 = c_{14} \frac{N}{m^2}.$$

If we can choose $|\varrho| = 1$ so that the quotient differ from 1 for all the other quadruples $(\eta_1, \eta_2, \eta_3, \eta_4)$ then just as in the case of (4) we get that the contribution of all the other terms does not exceed a bound depending only on m which can be included in the estimation $c_{11} \frac{N}{m^2}$ for large enough N . So we have but to choose ϱ and it

is here that we use the fact that $T(w)$ and consequently its inverse $S(z)$ is not a rotation. Suppose that for fixed $\eta_1, \eta_2, \eta_3, \eta_4$ the quotient in question is 1 that is

$$S(\varrho^{-1}\eta_1)S(\varrho^{-1}\eta_2) = S(\varrho^{-1}\eta_3)S(\varrho^{-1}\eta_4).$$

Varying ϱ on the circle $|\varrho|=1$ this can happen only for a finite number of ϱ since otherwise, by the analyticity of $S(z)$ it would follow

$$S(z\eta_1)S(z\eta_2) = S(z\eta_3)S(z\eta_4)$$

everywhere. The left hand side has zeros at $-\zeta e^{i\theta}\eta_1^{-1}$ and $-\zeta e^{i\theta}\eta_2^{-1}$ and only there (see (6)), the right hand side at $-\zeta e^{i\theta}\eta_3^{-1}$, $-\zeta e^{i\theta}\eta_4^{-1}$. Since the zeros of the left and right hand side have to coincide and $\zeta \neq 0$ this implies $\eta_1 = \eta_3$, $\eta_2 = \eta_4$ or $\eta_1 = \eta_4$, $\eta_2 = \eta_3$ which cases are now excluded. So there are only a finite number of „wrong“ ϱ for each quadruplex $(\eta_1, \eta_2, \eta_3, \eta_4)$. But to each m there are only a finite number of quadruples while there are only denumerably many m . This means at most denumerably many „wrong“ ϱ and we can choose a ϱ from the remaining part of $|\varrho|=1$ which makes all the quotients in question different from 1.

Now follows the actual construction. We use some notations and the first result from the proof of Theorem 1. Let s be a fixed integer definitely greater than $\frac{q}{p}$.

Successively we can select numbers N_m with the following properties:

N_m is a power of s , $N_{m+1} > N_m$,

$N_m \equiv qN_0(m)$, $N_0(m)$ as defined in (3).

Further, since

$$\sum_{l=0}^{\infty} s^l A_{s^l} \equiv \sum_{l=0}^{\infty} \frac{1}{s-1} \sum_{k=s^l}^{s^{l+1}-1} A_k = \frac{1}{s-1} \sum_{k=1}^{\infty} A_k = +\infty,$$

choosing N_{m+1} in the $(m+1)$ th step sufficiently large we can achieve that in addition

$$\sum_{N_m < s^l \leq N_{m+1}} s^l A_{s^l} \equiv \frac{1}{\sqrt{m}}$$

so that if

$$(7) \quad d_m \sum_{N_m < s^l \leq N_{m+1}} s^l A_{s^l} = \frac{1}{\sqrt{m}},$$

then $0 < d_m \leq 1$.

After these preparations we define a_k as follows. Let $k \in (N_m, N_{m+1}]$. The endpoints of this interval are powers of s and possibly the interval is subdivided by other powers of s . Suppose that k belongs to the subinterval $(s^{l-1}, s^l]$. To define a_k for such k we use our test function corresponding to m and the prescribed majorant A_n corresponding to s^l :

$$(8) \quad a_k \stackrel{\text{def}}{=} d_m A_{s^l} u_k^{(m)}.$$

If for $k \leq N_1$ we put $a_k = 0$, a_k is defined completely and we have to verify the three requirements of our theorem.

First of all $|a_k| \leq A_k$ is trivial by $0 < d_m \leq 1$, $|u_k^{(m)}| \leq 1$ (see definition (2) of $u_k^{(m)}$) and $A_{s^l} \leq A_k$ for $k \leq s^l$.

Let us turn to $\sum_{k=0}^{\infty} |a_k| < +\infty$.

$$\sum_{s^{l-1} < k \leq s^l} |a_k| = d_m A_{s^l} \sum_{s^{l-1} < k \leq s^l} |u_k^{(m)}| \leq d_m A_{s^l} \frac{s^l}{m}.$$

Here m is defined by $(s^{l-1}, s^l] \subset (N_m, N_{m+1}]$. Keeping m fixed and summing for all such intervals $(s^{l-1}, s^l]$

$$\sum_{N_m < k \leq N_{m+1}} |a_k| \leq \frac{d_m}{m} \sum_{N_m < s^l \leq N_{m+1}} s^l A_{s^l} = \frac{1}{m} \frac{1}{\sqrt{m}} = \frac{1}{m^{3/2}}$$

owing to (7) and summation for m gives finally

$$\sum_{k=0}^{\infty} |a_k| \leq \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} < +\infty.$$

Let us regard now the transformed series with coefficients

$$b_n = \sum_{k=0}^{\infty} t_{kn} a_k.$$

Recalling (1) we see that similarly as it was used for Theorem 1, in the expression of b_n we can change a_k for $k \leq pn$ and $k \geq qn$ anyhow, being careful only that the changed values should not exceed a fixed bound. The error made by this change is namely $O(e^{-c_2 n})$ and denoting the modified b_n by b_n^* it is sufficient to prove that

$$\sum_{n=0}^{\infty} |b_n^*| = +\infty.$$

Let $\frac{s^{l-1}}{p} < n \leq \frac{s^l}{q}$. This is possible by the choice $s > \frac{q}{p}$ of s . For $s^{l-1} < k \leq s^l$ a_k was defined by (8) and since for our n $s^{l-1} < pn$, $s^l \geq qn$ we change a_k in the way said before if we put in place of a_k (8) with the same l and m even for $k \leq s^{l-1}$ and $k > s^l$. (A bound of the changed values is provided by $|d_m A_{s^l} u_k^{(m)}| \leq A_0$). This means that we put

$$b_n^* = d_m A_{s^l} \sum_{k=0}^{\infty} t_{kn} u_k^{(m)} = d_m A_{s^l} v_n^{(m)}$$

recalling that the relation between $u_k^{(m)}$ and $v_n^{(m)}$ is the same as between a_k and b_n .

Using (3) with $N = \frac{s^l}{q} \left(\geq \frac{N_m}{q} \geq N_0(m) \right)$, $\alpha = \frac{q}{ps}$

$$\sum_{\frac{s^{l-1}}{p} < n \leq \frac{s^l}{q}} |b_n^*| = d_m A_{s^l} \sum_{\frac{s^{l-1}}{p} < n \leq \frac{s^l}{q}} |v_n^{(m)}| \leq c_9 d_m A_{s^l} \frac{s^l}{q \sqrt{m}} = \frac{c_{15}}{\sqrt{m}} d_m A_{s^l} s^l.$$

Summing for all l with $(s^{l-1}, s^l] \subset (N_m, N_{m+1}]$ and taking into account that the intervals $\left(\frac{s^{l-1}}{p}, \frac{s^l}{q} \right]$ are disjoint ($p < q$)

$$\sum_{\substack{N_m < n \leq N_{m+1} \\ p < n \leq \frac{s^l}{q}}} |b_n^*| \leq c_{15} \frac{1}{\sqrt{m}} d_m \sum_{N_m < s^l \leq N_{m+1}} s^l A_{s^l} = c_{15} \frac{1}{\sqrt{m}} \frac{1}{\sqrt{m}} = \frac{c_{15}}{m}$$

owing to (7). Summing for m

$$\sum_{n=0}^{\infty} |b_n^*| \cong c_{15} \sum_{m=1}^{\infty} \frac{1}{m} = +\infty,$$

q.e.d.

Reference

- [1] ALPÁR, L., Sur certaines transformées des séries de puissance absolument convergentes sur la frontière de leur cercle de convergence, *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** (1962), 287—316.

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