

## On the total curvature of closed curves in Riemannian manifolds

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Several proofs are known for the following theorem of W. FENCHEL: If  $\kappa(G)$  is the total curvature of a closed curve  $G$  in a euclidean space, then  $\kappa(G) \geq 2\pi$ , and the equality holds if and only if  $G$  is a convex plane curve.<sup>1)</sup> In this paper we show that the basic idea of one of the proofs of the above theorem<sup>2)</sup> can be applied to solve the corresponding problem for curves in Riemannian manifolds. The following theorem is proved: *If  $M$  is a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature, and  $\kappa(G)$  is the total curvature of a closed curve  $G$  in  $M$ , then  $\kappa(G) \geq 2\pi$ ; the equality holds if and only if  $G$  is the boundary of a 2-dimensional totally geodesic submanifold isometric with a convex domain of the euclidean plane.* Dropping either of the assumptions that  $M$  is simply connected and that it has everywhere nonpositive sectional curvature would not leave the lower bound  $2\pi$  valid, as a closed geodesic of a cylinder and a sufficiently small geodesic triangle in case of positive sectional curvature show. For convenience  $M$  is assumed to be of class  $C^\infty$  throughout the whole paper.

### 1. Riemannian manifolds with everywhere nonpositive sectional curvature

Let  $M$  be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature. The following facts are wellknown:

1. If  $m \in M$ , then  $\exp_m: M_m \rightarrow M$  is a diffeomorphism of the tangent space  $M_m$  onto  $M$ .

2. If  $p, q \in M$ , then there is exactly one geodesic  $\sigma_{pq}: [0, \alpha] \rightarrow M$  with  $\sigma_{pq}(0) = p$ ,  $\sigma_{pq}(\alpha) = q$ .

3. If  $\sigma: [0, \alpha] \rightarrow M$  is a geodesic and  $v \in M_{\sigma(x)}$ , then there is exactly one Jacobi field  $V: [0, \alpha] \rightarrow TM$  along  $\sigma$  with  $V(0) = 0$  and  $V(\alpha) = v$ .

The following lemma is a special case of Rauch's comparison theorem.<sup>3)</sup>

**Lemma 1.1.** *Let  $M$  be a complete Riemannian manifold with everywhere nonpositive sectional curvature,  $\sigma: [0, \alpha] \rightarrow M$  a geodesic,  $\sigma(0) = m$ ,  $v \in M_{\sigma(x)}$ ,  $x \in T(M_m)$*

<sup>1)</sup> FENCHEL [3].

<sup>2)</sup> BORSUK [2], MILNOR [4].

<sup>3)</sup> BISHOP—CRITTENDEN [1], 177—179. Conditions as to the equality are not explicitly stated there, but they are implicitly contained in the proof.

and  $v = d \exp_m x$ . Let further  $V: [0, \alpha] \rightarrow TM$  be the Jacobi field along  $\sigma$  with  $V(0) = 0$  and  $V(\alpha) = v$  and  $X: [0, \alpha] \rightarrow T(M_m)$  the field defined by  $V(t) = d \exp_m X(t)$ ,  $t \in [0, \alpha]$ . Then  $\|v\| \cong \|x\|$ , and in order the equality  $\|v\| = \|x\|$  to be valid the following conditions are necessary if  $v \neq 0$ :

- $\|V(t)\| = \|X(t)\|$ ,  $t \in [0, \alpha]$ .
- $\frac{V(t)}{\|V(t)\|}$ ,  $t \in (0, \alpha]$  is a parallel field along  $\sigma$ .
- The Riemannian curvature of the plane section spanned by the tangent vector  $\sigma_*(t)$  and  $V(t)$  is zero for  $t \in (0, \alpha]$ .

The length of a curve  $\varphi$  is denoted by  $L(\varphi)$  and the set of its points by  $|\varphi|$ . Let  $p, q, r$  be points of a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature and  $\sigma_{pq}, \sigma_{qr}, \sigma_{rp}$  the geodesics joining them. The set  $T_{pqr} = |\sigma_{pq}| \cup |\sigma_{qr}| \cup |\sigma_{rp}|$  is called a *geodesic triangle* and the set  $\Delta_{qr}^p = \{x: x \in |\sigma_{pq}|, z \in |\sigma_{qr}|\}$  a *geodesic triangular cell*. In general  $\Delta_{pr}^q, \Delta_{qr}^p, \Delta_{pq}^r$  are different. By a *euclidean representation* of  $T_{pqr}$  a point triple  $\{P, Q, R\}$  of a euclidean plane is meant for which  $\overline{PQ} = L(\sigma_{pq}), \overline{QR} = L(\sigma_{qr}), \overline{RP} = L(\sigma_{rp})$  hold.

The following lemma is a special case of one of A. D. Alexandrov's comparison theorems.

**Lemma 1.2.** *Let  $M$  be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature. If  $p, q, r \in M$  are different and  $\{P, Q, R\}$  is a euclidean representation of  $T_{pqr}$ , then the angle of the geodesics  $\sigma_{pq}, \sigma_{pr}$  is not greater than  $\angle PQR$ ; the two angles are equal if and only if  $\Delta_{pr}^q = \Delta_{qr}^p = \Delta_{pq}^r$  and this geodesic triangular cell is totally geodesic and isometric with the triangular domain  $PQR \Delta$ .*

A proof of this lemma can evidently be given by the method of Y. Tsukamoto<sup>4)</sup> on the basis of Lemma 1.1.

## 2. Concepts for the total curvature of curves in Riemannian manifolds

Let  $a_0, a_1, \dots, a_m, m \geq 2$  be such points of a Riemannian manifold  $M$  that  $a_i, a_{i+1}$  are different and can be joined by a unique minimal geodesic  $\sigma_i, i = 0, 1, \dots, m-1$ ; then the ordered set  $P = \{a_0, a_1, \dots, a_m\}$  is called a *geodesic polygon*. The points  $a_i$  are called the *vertices* of  $P$  and the geodesics  $\sigma_i$  its *sides*. The polygon is said to be *closed* or *open* according to whether  $a_m = a_0$ , or  $a_m \neq a_0$ . In case of a closed polygon  $a_{-1} = a_{m-1}, a_{m+1} = a_1$  are defined too. The angle  $\gamma_i$  of the geodesics  $\sigma_{i-1}, \sigma_i$  is called the *angle* of  $P$  at  $a_i$ , where  $i = 1, \dots, m-1$  if  $P$  is open, and  $i = 1, \dots, m$  when it is closed. By the *total curvature*  $\kappa(P)$  of  $P$  the sum of its angles is meant.

Let  $\varphi: [0, \alpha] \rightarrow M$  be a representation of a continuous curve  $G$  of  $M$  and  $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha, m \geq 2$  such a subdivision of  $[0, \alpha]$  that  $\{\varphi(\tau_0), \varphi(\tau_1), \dots, \varphi(\tau_m)\}$  is a geodesic polygon  $P$ , then  $P$  is said to be *inscribed* in  $G$ . If  $\{P_i\}_{i=1,2,\dots}$  is such a sequence of geodesic polygons inscribed in  $G$  that the maximal length

<sup>4)</sup> TSUKAMOTO [5].

of the sides of  $P_l$  tends to zero as  $l \rightarrow \infty$ , then it is called an *approximating sequence*. If for any approximating sequence  $\{P_l\}_{l=1,2,\dots}$  of geodesic polygons inscribed in  $G$  the corresponding sequence  $\{\kappa(P_l)\}_{l=1,2,\dots}$  is convergent, then  $G$  is said to have a total curvature, and  $\lim_{l \rightarrow \infty} \kappa(P_l)$ , which is then the same value for all approximating sequences, is called the *total curvature*  $\kappa(G)$  of  $G$ .

The total curvature of curves is usually defined as follows: Let  $G$  be a curve of class  $C^2$  in  $M$ ,  $\varphi: [0, \alpha] \rightarrow M$  the representation of  $G$  in terms of arc length and  $\kappa(s)$  the first curvature of  $G$  in the point  $\varphi(s)$ ,  $0 \leq s \leq \alpha$ , then  $\bar{\kappa}(G) = \int_0^\alpha \kappa(s) ds$  is called the total curvature of  $G$ . A proof is required therefore that if  $G$  is of class  $C^2$  then  $\kappa(G)$  exists and is equal to  $\bar{\kappa}(G)$ .

Let  $G$  be a curve of class  $C^1$  in  $M$  and  $\varphi: [0, \alpha] \rightarrow M$  the representation of  $G$  in terms of arc length. Denote by  $\varphi_*(t', t'')$  the vector obtained by parallel translation of the tangent vector  $\varphi_*(t')$  along  $\varphi$  to the point  $\varphi(t'')$  for  $t', t'' \in [0, \alpha]$ . Let  $\gamma(t', t'')$  be the angle of the vectors  $\varphi_*(t'), \varphi_*(t'')$ , then  $\gamma(t', t'')$ ,  $0 \leq t' \leq t'' \leq \alpha$  defines an interval function; if it is integrable, then  $\int_0^\alpha \gamma(t', t'')$  is denoted by  $\kappa^0(G)$ .

**Lemma 2.1.** *Let  $G$  be a curve of class  $C^2$  in a Riemannian manifold, then  $\kappa^0(G) = \bar{\kappa}(G)$ .*

Since

$$\lim_{t', t'' \rightarrow t} \frac{\gamma(t', t'')}{t'' - t'} = \lim_{t', t'' \rightarrow t} \frac{\|\varphi_*(t') - \varphi_*(t'')\|}{t'' - t'} = \|\nabla_{\varphi^*(t)} \varphi_*(t)\| = \kappa(t)$$

holds, the interval function  $\gamma(t', t'')$  is differentiable and  $\kappa(t)$ ,  $0 \leq t \leq \alpha$  is its derivative. Consequently the integrability of  $\gamma(t', t'')$  and  $\kappa^0(G) = \bar{\kappa}(G)$  are implied by standard theorems.

**Lemma 2.2.** *If  $G$  is a curve of class  $C^2$  in a Riemannian manifold  $M$ , then  $\kappa(G)$  exists and it is equal to  $\kappa^0(G)$ .*

Let  $\varphi: [0, \alpha] \rightarrow M$  be the representation of  $G$  in terms of arc length, and  $\delta > 0$ , then  $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha$  is called a  $\delta$ -subdivision of  $[0, \alpha]$ , if  $\tau_k - \tau_{k-1} \leq \delta$  for  $k = 1, \dots, m$ . To prove the lemma it suffices to show that to any  $\varepsilon > 0$  there is such a  $\delta > 0$ , that for any  $\delta$ -subdivision  $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha$ ,  $m \geq 2$  of  $[0, \alpha]$  the geodesic polygon  $P = \{\varphi(\tau_0), \varphi(\tau_1), \dots, \varphi(\tau_m)\}$  exists and

$$|\kappa(P) - \sum_{k=1}^{m-1} \gamma(\tau_{k-1}, \tau_k^0)| < \varepsilon$$

holds, where  $\tau_k^0 = \frac{\tau_k + \tau_{k+1}}{2}$ ,  $k = 0, 1, \dots, m-1$ .

It is enough to consider the case when  $G$  has no multiple points. A coordinate system  $v: U \rightarrow R^n$  of class  $C^2$  exists therefore, which is defined on a compact neighborhood  $U$  of  $|\varphi|$  and such that the coefficients of the Riemannian connexion of  $M$  all vanish when calculated in  $v$  for points of  $G$ . Let  $g_{ij}(x)$ ,  $i, j = 1, \dots, n$ ,  $x \in U$  be the components of the fundamental tensor in  $v$  and  $S = \sup \{|g_{ij}(x)|: i, j =$

$=1, \dots, n, x \in U\}$ . Let  $\varphi^i(t)$   $i=1, \dots, n, t \in [0, \alpha]$  denote the coordinates of  $\varphi(t)$  in  $v$ . Since  $G$  is of class  $C^2$ , to any  $\varepsilon > 0$  there is such a  $\delta_1 > 0$ , that

$$\left| \frac{\varphi^i(t'') - \varphi^i(t')}{t'' - t'} - \dot{\varphi}^i(t^0) \right| < \frac{\varepsilon}{8\alpha n \sqrt{S}} |t'' - t'|,$$

if

$$0 \leq t', t'' \leq \alpha, 0 < |t'' - t'| \leq \delta_1, t^0 = \frac{t' + t''}{2}.$$

A map  $\psi: [0, \xi] \rightarrow U$  is called a  $v$ -coordinate segment if  $v \circ \psi$  is linear. The sequence  $\psi_k: [0, \xi_k] \rightarrow U, k=0, 1, \dots, m-1, m \geq 2$  of  $v$ -coordinate segments is called a  $v$ -coordinate polygon if  $\psi_{k-1}(\xi_{k-1}) = \psi_k(0)$  for  $k=1, \dots, m-1$ . The  $v$ -coordinate segments  $\psi_k, k=0, 1, \dots, m-1$  are called *sides* and their endpoints *vertices* of the polygon. There exists such a  $\delta_2 > 0$ , that for any  $\delta_2$ -subdivision  $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha, m \geq 2$  the  $v$ -coordinate polygon with the consecutive vertices  $\varphi(\tau_0), \varphi(\tau_1), \dots, \varphi(\tau_m)$  exists. Consider a  $\zeta$ -subdivision  $0 = \tau_0 < \tau_1 < \dots < \tau_m = \alpha, m \geq 2$  with  $0 < \zeta < \delta_1, \delta_2$  and the corresponding  $v$ -coordinate polygon  $\psi_k: [0, 1] \rightarrow U, k=0, 1, \dots, m-1$  with the consecutive vertices  $\varphi(\tau_0), \varphi(\tau_1), \dots, \varphi(\tau_m)$ . Let  $\bar{\gamma}_k$  be the angle of the sides  $\psi_{k-1}, \psi_k$ , as well as  $\gamma'_k$  the angle of the vectors  $\psi_{k-1*}(1), \varphi_*(\tau_{k-1}^0, \tau_k)$  and  $\gamma''_k$  the angle of the vectors  $\psi_{k*}(0), \varphi_*(\tau_k^0, \tau_k)$  for  $k=1, \dots, m-1$ . Put  $\lambda^i = \frac{\varphi^i(\tau_k) - \varphi^i(\tau_{k-1})}{\tau_k - \tau_{k-1}}$  for  $i=1, \dots, n, k=1, \dots, m$ . In consequence to assertions above

$$\begin{aligned} \sum_{k=1}^{m-1} (\sin \gamma'_k + \sin \gamma''_k) &\leq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k)) (\lambda_k^i - \dot{\varphi}^i(\tau_{k-1}^0)) (\lambda_k^j - \dot{\varphi}^j(\tau_{k-1}^0))} + \right. \\ &\quad \left. + \sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k)) (\lambda_{k+1}^i - \dot{\varphi}^i(\tau_k^0)) (\lambda_{k+1}^j - \dot{\varphi}^j(\tau_k^0))} \right) \leq \\ &\leq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^n |S| \left( \frac{\varepsilon}{8\alpha n \sqrt{S}} (\tau_k - \tau_{k-1}) \right)^2} + \sqrt{\sum_{i,j=1}^n |S| \left( \frac{\varepsilon}{8\alpha n \sqrt{S}} (\tau_{k+1} - \tau_k) \right)^2} \right) < \frac{\varepsilon}{4}. \end{aligned}$$

By the assumptions of the lemma there is such a  $\delta_3 > 0$ , that for any  $\delta_3$ -subdivision of  $[0, \alpha]$  the inequalities  $\gamma'_k, \gamma''_k < \pi/2, k=1, \dots, m-1$  hold. Therefore

$$\begin{aligned} \left| \sum_{k=1}^{m-1} \bar{\gamma}_k - \sum_{k=1}^{m-1} \gamma(\tau_{k-1}^0, \tau_k^0) \right| &\leq \sum_{k=1}^{m-1} |\bar{\gamma}_k - \gamma(\tau_{k-1}^0, \tau_k^0)| \leq \sum_{k=1}^{m-1} (\gamma'_k + \gamma''_k) \leq \\ &\leq 2 \sum_{k=1}^{m-1} (\sin \gamma'_k + \sin \gamma''_k) < \frac{\varepsilon}{2} \end{aligned}$$

for any  $\zeta$ -subdivision of  $[0, \alpha]$  with  $0 < \zeta < \delta_1, \delta_2, \delta_3$ .

Let  $\varrho$  be the distance function of  $M$ . There is a neighbourhood  $V \subset U$  of  $|\varphi|$  and an  $\eta > 0$  such that if  $x, y \in V$  and  $\varrho(x, y) = \beta < \eta$ , then a unique minimal geodesic  $\sigma: [0, \beta] \rightarrow U$  and the  $v$ -coordinate segment  $\psi: [0, \beta] \rightarrow U$  both joining  $x, y$  exist. <sup>5)</sup>

<sup>5)</sup> BISHOP—CRITTENDEN [1], 246—250.

Let  $\dot{\sigma}^i(t)$  and  $\dot{\psi}^i(t)$  denote for  $i=1, \dots, n$ ,  $t \in [0, \beta]$  the coordinates of the vectors  $\sigma_*(t)$ ,  $\psi_*(t)$  respectively, then

$$|\dot{\sigma}^i(z) - \dot{\psi}^i(z) - [\dot{\sigma}^i(t) - \dot{\psi}^i(t)]| \leq \int_0^\beta \sum_{k,l=1}^n |\Gamma_{kl}^i \dot{\sigma}^k(\tau) \dot{\sigma}^l(\tau)| d\tau$$

for  $0 \leq t \leq \beta$ ,  $z=0, \beta$ . There is a  $t_i$  with  $0 \leq t_i \leq \beta$  for every  $i=1, \dots, n$ , such that  $\dot{\sigma}^i(t_i) - \dot{\psi}^i(t_i) = 0$ . The lengths of the geodesic polygons inscribed in  $G$  have an upper bound  $L < \infty$ . The absolute values of coordinates of tangent vectors to geodesics in points of  $U$  have an upper bound  $N < \infty$ . There is a neighbourhood  $W \subset V$

of  $|\varphi|$  such that for points in  $W$  the inequalities  $|\Gamma_{kl}^i| \leq \frac{\varepsilon}{8n^3 \sqrt{SLN^2}}$ ,  $i, k, l=1, \dots, n$  hold. Let  $x, y \in |\varphi|$  be such points that  $\varrho(x, y) = \beta < \eta$ , and  $|\sigma| \subset W$ , then by the preceding assertions  $|\dot{\sigma}^i(z) - \dot{\psi}^i(z)| \leq \frac{\varepsilon}{8n\sqrt{SL}} \beta$ ,  $i=1, \dots, n$ . There is a  $\delta_4 > 0$  such that for

any  $\delta_4$ -subdivision of  $[0, \alpha]$  the corresponding geodesic and  $v$ -coordinate polygons lie in  $W$  and the distances of the consecutive vertices are less than  $\eta$ . Denote  $\sigma_k: [0, \beta_k] \rightarrow W$ ,  $\psi_k: [0, \beta_k] \rightarrow W$ ,  $k=0, 1, \dots, m-1$  the sides of these polygons. Let  $\bar{\gamma}_k$  be the angle of the vectors  $\sigma_{k-1*}(\beta_{k-1})$ ,  $\psi_{k-1*}(\beta_{k-1})$  and  $\bar{\gamma}_k''$  that of  $\sigma_{k*}(0)$ ,  $\psi_{k*}(0)$  for  $k=1, \dots, m-1$ . By the assertions above

$$\begin{aligned} & \sum_{k=1}^{m-1} (\sin \bar{\gamma}_k' + \sin \bar{\gamma}_k'') \leq \\ & \leq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k)) (\dot{\sigma}_{k-1}^i(\beta_{k-1}) - \dot{\psi}_{k-1}^i(\beta_{k-1})) (\dot{\sigma}_{k-1}^j(\beta_{k-1}) - \dot{\psi}_{k-1}^j(\beta_{k-1}))} + \right. \\ & \quad \left. + \sqrt{\sum_{i,j=1}^n g_{ij}(\varphi(\tau_k)) (\dot{\sigma}_k^i(0) - \dot{\psi}_k^i(0)) (\dot{\sigma}_k^j(0) - \dot{\psi}_k^j(0))} \right) \leq \\ & \leq \sum_{k=1}^{m-1} \left( \sqrt{\sum_{i,j=1}^n |S| \left( \frac{\varepsilon}{8n\sqrt{SL}} \varrho(\varphi(\tau_{k-1}), \varphi(\tau_k)) \right)^2} + \right. \\ & \quad \left. + \sqrt{\sum_{i,j=1}^n |S| \left( \frac{\varepsilon}{8n\sqrt{SL}} \varrho(\varphi(\tau_k), \varphi(\tau_{k+1})) \right)^2} \right) < \frac{\varepsilon}{4}. \end{aligned}$$

By the assumptions of the lemma there is a  $\delta_5 > 0$  such that for any  $\delta_5$ -subdivision of  $[0, \alpha]$  the inequalities  $\bar{\gamma}_k', \bar{\gamma}_k'' < \frac{\pi}{2}$ ,  $k=1, \dots, m-1$  hold. Consequently for any  $\zeta$ -subdivision, with  $0 < \zeta < \delta_4, \delta_5$  the inequalities

$$\left| \sum_{k=1}^{m-1} (\bar{\gamma}_k - \gamma_k) \right| \leq \sum_{k=1}^{m-1} |\bar{\gamma}_k - \gamma_k| \leq \sum_{k=1}^{m-1} (\bar{\gamma}_k' + \bar{\gamma}_k'') \leq 2 \sum_{k=1}^{m-1} (\sin \bar{\gamma}_k' + \sin \bar{\gamma}_k'') < \frac{\varepsilon}{2}$$

are valid. Let  $0 < \delta < \delta_j$  be for  $j=1, \dots, 5$ , then

$$\left| \kappa(P) - \sum_{k=1}^{m-1} \gamma(\tau_{k-1}^0, \tau_k^0) \right| \leq \left| \sum_{k=1}^{m-1} \gamma_k - \bar{\gamma}_k \right| + \left| \sum_{k=1}^{m-1} \bar{\gamma}_k - \sum_{k=1}^{m-1} \gamma(\tau_{k-1}^0, \tau_k^0) \right| < \varepsilon$$

for any  $\delta$ -subdivision of  $[0, \alpha]$ .

### 3. The total curvature of closed curves in Riemannian manifolds with everywhere nonpositive sectional curvature

In what follows  $M$  is assumed to be a complete simply connected Riemannian manifold with everywhere nonpositive sectional curvature.

**Lemma 3. 1.** *Let  $P = \{a_0, \dots, a_{i-1}, a_i, \dots, a_m\}$  and  $P' = \{a_0, \dots, a_{i-1}, a, a_i, \dots, a_m\}$ ,  $m \geq 2$  be closed geodesic polygons of  $M$  with sides  $\sigma_k: [0, \alpha_k] \rightarrow M$ ,  $k=0, 1, \dots, m-1$  and  $\sigma_k: [0, \alpha_k] \rightarrow M$ ,  $k=0, 1, \dots, i-2$ ,  $\sigma'_{i-1}: [0, \alpha'_{i-1}] \rightarrow M$ ,  $\sigma: [0, \alpha] \rightarrow M$ ,  $\sigma_k: [0, \alpha_k] \rightarrow M$ ,  $k=i, \dots, m-1$ , then  $\kappa(P) \leq \kappa(P')$ ; the equality holds if and only if  $\Delta_{a_{i-1}a_i}^a$  is a totally geodesic triangular cell isometric with one in the euclidean plane, and  $\sigma'_{i-1*}(0) = u\sigma_{i-2*}(\alpha_{i-2}) + v\sigma_{i-1*}(0)$ ,  $\sigma_*(\alpha) = u'\sigma_{i-1*}(\alpha_{i-1}) + v'\sigma_{i*}(0)$  with  $0 \leq u, v, u', v'.$*

Let  $\gamma_0, \gamma_1, \dots, \gamma_{i-1}, \gamma_i, \dots, \gamma_{m-1}$  denote the angles of  $P$  and  $\gamma_0, \gamma_1, \dots, \gamma'_{i-1}, \gamma, \gamma_i, \dots, \gamma_{m-1}$  those of  $P'$ . Let  $\xi$  be the angle of the vectors  $\sigma'_{i-1*}(0), \sigma_{i-1*}(0)$  and  $\eta$  that of the vectors  $\sigma_*(\alpha), \sigma_{i-1*}(\alpha_{i-1})$ , then  $\kappa(P') = \kappa(P) + \gamma'_{i-1} - \gamma_{i-1} + \gamma + \gamma_i - \gamma_i \geq \kappa(P) + \xi + \gamma'_{i-1} - \gamma_{i-1} + \eta + \gamma_i - \gamma_i \geq \kappa(P)$  by Lemma. 1. 2. It can be verified by the same lemma that the above condition of equality is necessary and sufficient.

**Lemma 3. 2.** *Let  $P = \{a_0, a_1, \dots, a_m\}$ ,  $m \geq 2$  be a closed geodesic polygon of  $M$ , then  $\kappa(P) \geq 2\pi$ ; the equality holds if and only if  $P$  is the boundary of a totally geodesic submanifold isometric with a convex polygonal domain of the euclidean plane.*

Consider the closed geodesic polygons  $P_1 = \{a_0, a_{m-2}, a_{m-1}, a_m\}$ ,  $P_2 = \{a_0, a_{m-3}, a_{m-2}, a_{m-1}, a_m\}$ , ...,  $P_m = P$ , then by Lemmas 1. 2 and 3. 1  $2\pi \leq \kappa(P_1) \leq \kappa(P_2) \leq \dots \leq \kappa(P)$ . Assume that  $\kappa(P) = 2\pi$ , then by an obvious mathematical induction and by the same lemmas it can be shown that

$$[P] = \Delta_{a_0 a_{m-1}}^{a_{m-2}} \cup \Delta_{a_0 a_{m-2}}^{a_{m-3}} \cup \dots \cup \Delta_{a_0 a_2}^{a_1}$$

is a totally geodesic 2-dimensional submanifold with boundary  $P$  and mapped isometrically by  $\exp_{a_0}^{-1}$  onto a convex polygonal domain in  $M_{a_0}$ . The submanifold  $[P]$  is unique.

**Theorem.** *Let  $G$  be a closed curve of class  $C^2$  in a complete simply connected Riemannian manifold  $M$  with everywhere nonpositive sectional curvature, and  $\kappa(G)$  the total curvature of  $G$ , then  $\kappa(G) \geq 2\pi$ ; equality holds if and only if  $G$  is the boundary of a totally geodesic submanifold of  $M$  isometric with a convex domain of the euclidean plane.*

There is an approximating sequence  $\{P_l\}_{l=1,2,\dots}$  of closed geodesic polygons inscribed in  $G$ , such that  $P_l, P_{l+1}$   $l=1, 2, \dots$  are in the same relation as  $P, P'$  of Lemma 3. 1, and  $a \in |\varphi|$  is a common vertex of all of them. Consequently  $2\pi \leq \kappa(P) \leq \kappa(P_{l+1})$ , and  $\kappa(G) = \lim_{l \rightarrow \infty} \kappa(P_l) \geq 2\pi$ . Assume that  $\kappa(G) = 2\pi$ , then  $\kappa(P_l) = 2\pi$ ,  $l=1, 2, \dots$ . Let  $[P_l]$  denote the totally geodesic submanifold bounded by  $P_l$ . By Lemmas 1. 2, 3. 1, 3. 2 one can show that  $\exp_a^{-1}(\bigcup_{l=1} [P_l])$  is a convex plane domain mapped isometrically by  $\exp_a$  onto a totally geodesic submanifold bounded by  $G$ .

## References

- [1] R. L. BISHOP—R. J. CRITTENDEN, *Geometry of manifolds*, New York, 1964.
- [2] K. BORSUK, Sur la courbure total des courbes fermées. *Ann. Soc. Polon. Math.*, **20** (1947), 251—265.
- [3] W. FENCHEL, Über Krümmung und Windung geschlossener Raumkurven, *Math. Ann.* **101** (1929), 238—252.
- [4] J. W. MILNOR, On the total curvature of knots, *Ann. of Math.*, **52** (1950), 248—257.
- [5] Y. TSUKAMOTO, On a theorem of A. D. Alexandrov, *Mem. Fac. Sci. Kyushu Univ. Ser. A.*, **15** (1962), 83—89.

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