

Note on Tauberian constants

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1. Introduction

We shall discuss the following problem: *there is given*

- (i) an ultimately monotonic positive sequence $\tau_k \nearrow \infty$;
 - (ii) a class $U(\tau_k)$ of series Σu_k of complex terms, and with partial sums $s_n = u_0 + u_1 + \dots + u_n$, satisfying the Tauberian condition $|\tau_k u_k| = O(1)$;
 - (iii) a regular summability method $\{a_n(t)\}$ transforming s_k into $\sigma_t = \Sigma a_n(t) s_n$, where $\{a_n(t)\}$ may represent a matrix $a_{m,n}$ when $a_n(t) = a_n([t])$;
 - (iv) a positive number q (not necessarily an integer);
- it is required to find the smallest constant $A(q)$, finite or infinite such that the inequality

$$(1.1) \quad \limsup_{t \rightarrow \infty, n/t \rightarrow q} |\sigma_t - s_n| \leq A(q) \limsup_{k \rightarrow \infty} |\tau_k u_k|$$

should be satisfied for every series in $U(\tau_k)$.

The constant $A(q)$ is called a Tauberian constant, in short a T -constant.

Many results are known for the Tauberian condition $|ku_k| = O(1)$. We know that for this class finite T -constants exist for every q for the Abel, Cesàro, Hausdorff, Quasi-Hausdorff and other methods [2, 4, 8, 10, 11, 14, etc.], and that $A(q)$ is unbounded as q tends to zero or to infinity.

Recent investigations of AGNEW, ANJANEYULU and MEIR [3, 5, 15] relate to the class $U(\sqrt{k})$. It has been found that with each method $a_n(t)$ a positive constant λ is associated such that $A(q)$ is finite if and only if $q = \lambda$ and n/t tends to λ „closely enough”. For the Borel method Agnew found $\lambda = 1$, and for the Laurent method Anjaneyulu found $\lambda = x/(1-x)$. The „closeness” is measured by the coupling relation $\omega = (n - \lambda t)/\sqrt{t}$, and $A(q)$ is found to be finite if and only if $\Omega = \limsup |\omega|$ is finite, and $A(\lambda)$ is then a function of Ω . Obviously $A(\lambda, \Omega)$ increases with Ω .

The object of this note is to show that there are only those two types of behaviour, depending on whether τ_k tends to infinity at the same rate as k or slower, this being true for any regular summability method which has a finite T -constant.

Added 27. 3. 1967. In two recent papers BIEGERT [27, 28] has studied the T -constants for the summability methods discussed in (4.1) of our paper. He obtained

* Sent on 18. 12. 1965, but lost in the post.

formulae when $\tau_k = k^p$ for all real p , and his values for the ratio λ agree with our values. (Biegert's formula (2.1) in [28] should be corrected to

$$(\gamma m/D_V - t)/(\gamma^{1-p} t^p) = \omega \quad \text{with} \quad \limsup_{t \rightarrow \infty} |\omega| < \infty,$$

as stated in recent letters. A similar correction should be made to his formula (16. 2) in [27]).

2. Notation and lemmas

We shall always assume that $n=n(\alpha)$ and $t=t(\alpha)$ are monotonic functions of a parameter $\alpha \geq 0$, both tending to infinity with α , $n(0)=t(0)=0$, $n(\alpha)$ being an integer for all α .

A positive sequence x_k will be called *ultimately monotonic* if it is monotonic for $k > k_0$. The suffix k will denote an integer; s_n is a step-function of α and σ_t is a function of α .

If x_k is ultimately monotonic decreasing, tending to zero, we shall write $x_k \searrow 0$. When $1/x_k \searrow 0$, we shall write $x_k \nearrow \infty$.

When x_k and y_k are positive and $x_k/y_k \searrow 0$, we shall write $x_k \ll y_k$. $U(\tau_k)$ will denote the *class* of series with complex terms Σu_k satisfying $|\tau_k u_k| = O(1)$. $A_\epsilon(q)$ or $A(q)$ will denote the constant in (1.1) for $U(\tau_k)$.

We shall also use the order notation „ $\ll$ ” for positive *functions* of α : for example $r_t \ll r_t^*$ will mean that both $r_{t(\alpha)}$ and $r_{t^*(\alpha)}^*$ are ultimately monotonic functions of α , and that $r_t/r_t^* \searrow 0$.

When $r_t \ll t$ and $n/t \rightarrow \lambda$ so closely that $n/t = \lambda + O(r_t/t)$, then the function $\omega(\alpha)$ defined by $\omega = (n - \lambda t)/r_t$ is bounded as $\alpha \rightarrow \infty$. Hence the non-negative constant $\Omega = \limsup |\omega(\alpha)|$ and the order of smallness of r_t are both measures of the closeness of approach of the ratio n/t to the limit λ . The smaller is r_t and the smaller is Ω , the closer comes n/t to λ . In lemma 2 we shall show that r_t can be replaced by another function ϱ_n of α .

Lemma 1. If $t/t^* \rightarrow 1$ as $\alpha \rightarrow \infty$, and if for some $c \geq 0$, $t^{-c} \ll r_t \ll t^c$, then $r_t/r_t^* \rightarrow 1$.

PROOF. For $\alpha > \alpha_0$, $t-1 < t^* < t+1$, hence if r_t is ultimately increasing, $(t-1)^c \leq r_{t-1}/r_t < r_{t^*}/r_t \leq (t+1)^c/t^c$. If r_t is ultimately decreasing, then the same argument can be applied to $1/r_t$.

Lemma 2. If, for some positive c , $n^{-c} \ll \varrho_n \ll n$, and if for a positive q we define $r_t = \varrho_{[qt]}$, $\omega = (n - qt)/\varrho_n$, and $\omega^* = (n - qt)/r_t$, then as $\alpha \rightarrow \infty$, $\limsup |\omega| = \limsup |\omega^*|$.

PROOF. Let $p = p(\alpha) = (|n - qt| + n^{-c})/\varrho_n$, $p^* = (|n - qt| + n^{-c})/r_t$, then p and p^* are positive, and $p/p^* = r_t/\varrho_n = \varrho_{[qt]}/\varrho_n \rightarrow 1$ by lemma 1, since $qt/n = 1 - \omega\varrho_n/n \rightarrow 1$ when ω is bounded. Also $p = |\omega| + n^{-c}/\varrho_n$, hence $\limsup p = \limsup |\omega|$, and similarly $\limsup p^* = \limsup |\omega^*|$. Again when ω is unbounded, so is $\omega^* = \omega\varrho_n/r_t$.

Lemma 3. If $b_k > 0$, $c_k \nearrow \infty$, then for $n_0 < n < p$,

$$c_n \sum_{k=n}^p b_k \leq \sum_{k=n}^p b_k c_k \leq c_p \sum_{k=n}^p b_k.$$

This is trivial, but will be often used.

In what follows we shall *define* for a positive number q , and for a positive sequence x_n such that $x_n \ll n$,

$$(2.1) \quad \omega = \omega(\alpha) = \frac{n(\alpha) - qt(\alpha)}{x_{n(\alpha)}}, \quad \text{and} \quad \Omega = \limsup_{\alpha \rightarrow \infty} |\omega(\alpha)|.$$

When $\tau_k \ll k$, the smallest constant A , finite or infinite, for which the inequality

$$(2.2) \quad \limsup_{\limsup |\omega| = \Omega} |\sigma_t - s_n| \leq A \limsup_{k \rightarrow \infty} |\tau_k u_k|$$

is satisfied for a *fixed* Ω and for every series in $U(\tau_k)$ will be denoted by

$$(2.3) \quad A(q, \Omega, x_n).$$

When $x_n = 1$ for $n > n_0$, we shall write $A(q, \Omega, 1_n)$.

3. Theorems and proofs

Theorem 3. I. *If a regular summability method $\{a_n(t)\}$ has for the class $U(k)$ a finite T -constant for some positive q , then the constant $A(q)$ is finite for every positive q^* , and $A(q) \rightarrow \infty$ as $q \rightarrow 0$ and as $q \rightarrow \infty$.*

Theorem 3. II. *If $1_k \ll \tau_k \ll k$, $1_n \ll x_n \ll n$, and if for the class $U(\tau_k)$ and for the regular method $\{a_n(t)\}$ the T -constant $A(q, \Omega, x_n)$ is finite for one positive value λ of q , and for one non-negative value Ω_0 of Ω , then*

- (i) $A(q, \Omega, x_n) = \infty$ when $q \neq \lambda$ for every Ω and for any x_n ($1_n \ll x_n \ll n$);
- (ii) x_n/τ_n is bounded;
- (iii) $A(\lambda, \Omega, y_n)$ is finite for every y_n such that y_n/τ_n is bounded and for every Ω , it has its minimum at $\Omega = 0$, and tends to infinity with Ω ;
- (iv) if the same method is applied to another class $U(\tau_k^*)$, where $\tau_k \ll \tau_k^* \ll k$, then finite T -constants exist for the same λ , for all Ω and for all y_n such that y_n/τ_n^* is bounded.

Theorem 3. III. *There are regular methods*

- (i) *which have no finite T -constants for $U(k)$;*
- (ii) *which have finite T -constants for $U(k)$ but none for any class $U(\tau_k)$ such that $\tau_k \ll k$.*

PROOF OF 3. I. Let q and q_0 be distinct positive numbers, and let $A(q_0)$ be finite. Let $n/t \rightarrow q_0$, and $n/t' \rightarrow q$. For each t' we define n' as $n' = [q_0 t']$ so that $n'/t' \rightarrow q_0$ and $n'/n \rightarrow q_0/q$. Let Σu_k be any series in $U(k)$ so that $\limsup |ku_k| = L$ is finite. Here, and later we shall use the trivial inequalities:

$$(3.1) \quad |\sigma_{t'} - s_n| \leq |s_{n'} - s_n| + |\sigma_{t'} - s_{n'}|,$$

$$(3.2) \quad |\sigma_{t'} - s_n| \geq |s_{n'} - s_n| - |\sigma_{t'} - s_{n'}|.$$

As $\alpha \rightarrow \infty$, $\limsup |\sigma_{t'} - s_{n'}| \leq A(q_0)L$, and for $\alpha > \alpha_0$

$$|s_{n'} - s_n| \leq (L + \varepsilon) \left| \sum_{k=n}^{n'} 1/k \right| \leq (L + \varepsilon) |\log(q_0/q)|,$$

so that, by (3.1) and (3.2),

$$|\log(q_0/q)| - A(q_0) \leq A(q) \leq |\log(q_0/q)| + A(q_0).$$

This proves the theorem.

PROOF OF 3. II. To prove (i), we first observe that

$$(3.3) \quad \text{if } \Omega < \Omega_0 \text{ then } A(q, \Omega, x_n) \leq A(q, \Omega_0, x_n) \text{ for any } q \text{ and any } x_n,$$

since the left term refers to a closer approach. Let $q \neq \lambda$. For each n we define first n' as $n' = [\lambda n/q]$ and the $t' = n'/\lambda$ so that for the series $\Sigma 1/\tau_k$ we obtain $\limsup |\sigma_{t'} - s_{n'}| \leq A(\lambda, 0, x_n)$, and this is finite by (3.3). Again, for $n > n_0$, $n'/n = [\lambda n/q]/n > C > 0$, where C is a constant different from 1. Applying lemma 3 we obtain when $n' > n$

$$(3.4) \quad |s_{n'} - s_n| = \left| \sum_{k=n}^{n'} \frac{1}{k} \frac{k}{\tau_k} \right| > \left| \log \frac{n'}{n} \right| \frac{n}{\tau_n} > |\log C| \frac{n}{\tau_n},$$

and when $n' < n$, the last term is replaced by $|\log C| n'/\tau_{n'}$. Hence (i) follows from (3.2).

To prove (ii) and (iii) we consider any bounded function $\omega(x)$, such that $\limsup |\omega| = \Omega$, and any given sequence such that $1_n \ll x_n \ll n$. For each n we define n' as $n' = [n - \omega x_n]$ and t' as $t' = n'/\lambda$, so that for any series in $U(\tau_k)$ again $\limsup |\sigma_{t'} - s_{n'}| \leq A(\lambda, 0, x_n)L$. Let $\omega' = (n - \lambda t')/x_n = (n - n')/x_n$, then $\omega' = \omega + \eta_n/x_n$, where for $n > n_0$ $|\eta_n| < 1$, so that $\limsup |\omega'| = \limsup |\omega|$. Again, for $n > n_1$,

$$(3.5) \quad 1 - (\Omega + 1)x_n/n < n'/n < 1 + (\Omega + 1)x_n/n,$$

hence, for $n > n_2$

$$(3.6) \quad (\Omega + 1)x_n/2n < |\log(n'/n)| < 2(\Omega + 1)x_n/n.$$

Taking $u_k = 1/\tau_k$, we obtain, as in (3.4), for $n > n_3$ and $n' > n$,

$$(3.7) \quad |s_{n'} - s_n| > |\log(n'/n)| n/\tau_n > (\Omega + 1)x_n/2\tau_n,$$

and (ii) follows from (3.2) and (3.7) when $\Omega = \Omega_0$.

For any u_k such that $\limsup |\tau_k u_k| = L$ is finite, using (3.6) and lemma 3, and with y_n replacing x_n , and assuming that $n' > n$, we obtain

$$(3.8) \quad \begin{aligned} \limsup |s_{n'} - s_n| &< 2L(\Omega + 1)(y_n/n)(n'/\tau_{n'}) = \\ &= 2L(\Omega + 1)(n'/n)(\tau_n/\tau_{n'})(y_n/\tau_n). \end{aligned}$$

Now (3.5) shows that $n'/n \rightarrow 1$, and by lemma 1, $\tau_n/\tau_{n'} \rightarrow 1$ and y_n/τ_n is bounded, hence by (3.1) $A(\lambda, \Omega, y_n)$ is finite for any Ω . The same result holds when $n' < n$. The T -constant decreases with Ω , and exists at $\Omega = 0$, hence it attains its minimum there, and (3.7) shows that it tends to infinity with Ω . This proves (iii).

To prove (iv) we observe that $U(\tau_k^*)$ is a subclass of $U(\tau_k)$ so that $A_{\tau^*}(\lambda, \Omega_0, x_n)$ is finite, hence by (iii) so is $A_{\tau^*}(\lambda, \Omega, y_n)$ for every Ω and for every y_n such that y_n/τ_n^* is bounded.

PROOF OF 3. III. (i) follows from the example:

Let $\{a_n(t)\}$ be defined by $\sigma_t = s_{[t^2]}$, then for the series $\Sigma 1/k$ and for $q > 0$ we have $\sigma_t - s_n = s_{[n^2/q^2]} - s_n = 0$ ($\log n$) which is unbounded.

To prove (ii), let $0 < x < y < 1$, and let $L(x)$, $L(y)$ be the corresponding Laurent matrices. Anjaneyulu ([5]) proved that for $L(x)$ the value of λ is $x/(1-x)$, hence for $L(y)$ it is $\lambda' = y/(1-y)$. Let $\{a_{m,n}\}$ be the matrix formed by taking all rows in succession alternately from $L(x)$ and $L(y)$. Then for any $U(\tau_k)$ with $1_k \ll \tau_k \ll k$, theorem 3. II (i) would require that for odd m , $(m+1)/2n$ should tend to λ , and for even m , $m/2n$ should tend to λ' . This shows that no finite T -constant exists for such a class. But for $U(k)$ both $L(x)$ and $L(y)$ have finite T -constants for any positive q , $A(q)$ and $A'(q)$ say. Hence for our „mixed” matrix the T -constant is the larger of $A(q/2)$ and $A'(q/2)$.

4. Examples

Theorem 3. II has shown how the existence of a finite T -constant depends for a given method on the class $U(\tau_k)$ to which the method is applied. If $\tau_k \ll k$, there is at most one value of q , $q = \lambda$, to which the ratio n/t must tend, and the smaller is the order of τ_k , the closer must be the approach. To find finite T -constants to any given regular method, the first problem is to find the (unique) number λ (if any) belonging to the method, the next problem is to find the best possible, hence the smallest, order of τ_k , and the last problem is to find the values of the T -constants for all admissible values of q , Ω and x_n . Here we shall show how λ can be found for a wide class of methods; we do not attempt to give general rules for solving the other two more difficult problems

(4.1) *Sonnenschein methods* [20, 21].

These are generated by a non-constant function $a(z) = \sum a_n z^n$ such that $a_n \geq 0$, and $\sum a_n = 1$. If, for $t \geq 0$, $(a(z))^t = \sum a_n(t) z^n$, then the sequence to function transformation $\sigma_t = \sum_n a_n(t) z^n$ is regular. A *Sonnenschein matrix* is given by $a_{m,n} = a_n(m)$ for $n, m = 0, 1, 2, \dots$. For example:

e^{z-1} generates the *Borel method* $\{e^{-t} t^n / n!\}$ [6, 12];

$1 - p + pz$, $0 < p < 1$, the *Euler matrix* $\left\{ \binom{m}{n} (1-p)^{m-n} p^n \right\}$ [1, 16, 22, 25];

$\frac{pz}{1 - (1-p)z}$, $0 < p < 1$, the *Hardy—Littlewood—Fekete (circle) matrix*,

also called *Taylor matrix* $\left\{ \binom{n-1}{m-1} p^m (1-p)^{n-m} \right\}$

[7, 9, 12, 13, 16, 23, 24, 25, 26];

$\frac{p}{1 - (1-p)z}$, $0 < p < 1$, the *Laurent matrix* $\left\{ \binom{m+n-1}{m-1} p^m (1-p)^n \right\}$ [16, 25].

When $\sum a_n$ converges so that its sum is $a'(1)$ (as in all the above examples), then for $t \geq 1$ and $|z| \leq 1$, $\sum a_n(t) z^n = z d(a(z))^t / dz = t z (a(z))^{t-1} a'(z)$ so that $\sum a_n(t) = t a'(1)$. Hence the method $\{a_n(t)\}$ transforms the sequence $s_n = n$ into the function

$a'(1)t$. Hence if $n/t \rightarrow q$, then $\limsup |\sigma_t - s_n| = \limsup |a'(1)t - qt|$ is finite for this particular sequence if and only if $q = a'(1)$, so that $a'(1)$ is the value of λ . Hence for the: Borel method $\lambda = 1$, Euler matrix $\lambda = p$, Taylor matrix $\lambda = 1/p$, Laurent matrix $\lambda = (1-p)/p$, the same as given by Anjaneyulu ([5]), his parameter x being our parameter $1-p$.

(4. 2) Hausdorff and allied matrices.

The Hausdorff matrices are generated by a mass-function $\mu(x)$ of bounded variation in the closed interval $[0, 1]$, and such that $\mu(x)$ is continuous at $0+$, and $\mu(0)=0$, $\mu(1)=1$. The matrix is defined by the integral

$$a_{m,n} = \int_0^1 \binom{m}{n} (1-x)^{m-n} x^n d\mu(x), \quad [6, 12],$$

and the integrand is analogue to the Euler matrix.

The Quasi-Hausdorff matrix is generated by the same type of mass function, and

$$a_{m,n} = \int_0^1 \binom{n-1}{m-1} x^m (1-x)^{n-m} d\mu(x) \quad [12, 17, 18],$$

the integrand being analogue to the Taylor matrix.

The Laurent-Hausdorff matrix is generated by the same mass function

$$a_{m,n} = \int_0^1 \binom{m+n-1}{m-1} x^m (1-x)^n d\mu(x) \quad [19] \text{ analogue to the Laurent matrix.}$$

Hence by an argument similar to that in (4.1) we obtain:

$$\text{for the Hausdorff matrix } \lambda = \int_0^1 x d\mu(x),$$

$$\text{Quasi-Hausdorff matrix } \lambda = \int_0^1 \frac{d\mu(x)}{x},$$

$$\text{Laurent-Hausdorff matrix } \lambda = \int_0^1 \frac{(1-x) d\mu(x)}{x}.$$

(4. 3) Abel method.

This is a series to function transformation, defined by $\sigma_x = \sum u_k x^k$, $0 < x < 1$, and $x \nearrow 1$. Putting $x = e^{-1/t}$, we obtain $\sigma_t = \sum u_k e^{-k/t}$, $t \rightarrow \infty$. When $s_n = n+1$, so that $u_k = 1$, we have $\sigma_t = 1/(1 - e^{-1/t})$, hence $|\sigma_{n/q} - s_n|$ is bounded as $n \rightarrow \infty$ only if $q = 1$. Thus the only possible value for λ is 1.

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(Received June 26, 1967.)