

On compact objects in categories

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To Prof. A. G. Kuroš on his 60th birthday

Introduction

In the paper [6] we have introduced the concept of M -compact objects in a category. The M -compact objects correspond to the 'in a narrow sense linearly compact' modules introduced by LEPTIN [2] in the category of modules. M -compact objects take an important part in the characterisation of M -semi-simple objects (cf. [6]).

It is the purpose of this paper to derive some results concerning M -compact objects. In § 1 we establish assertions used in the subsequent sections. Some of them are of some interest, and complete the results of [6] concerning the inverse system belonging to an object.

In his paper [4] SULIŃSKI has defined the concept of M -representable ideals, and developed results concerning it. The aim of § 2 is to show a close relation between M -representable ideals and M -compact objects. It will turn out that in an M -compact object any ideal which contains the M -radical and which is an M -compact object, is M -representable and conversely, in an M -compact object any M -representable ideal is an M -compact object. Further, we obtain that any M -representable ideal of an M -compact M -semi simple object is a direct factor of this object.

In § 3 we introduce a closure operation on the subobjects of an object in a rather natural way and we prove that for M -compact objects this closure operation is topological. Thus M -compact objects can be considered as objects equipped with a topology. Making use of this closure operation we define dense subobjects of an object and also M -compactifications of an object. It will be proved that any M -semi-simple object has an M -compactification and can be embedded as a dense subobject in a direct product of M -objects.

§ 1. Preliminaries

Let C be a category. The objects and maps of C will be denoted by small Latin and small Greek letters, respectively. By definition C satisfies the following conditions

(C₁) If $\alpha: a \rightarrow b$ and $\beta: b \rightarrow c$ are maps, then there is a uniquely defined map $\alpha\beta: a \rightarrow c$ which is called the product of the maps α and β ;

(C₂) If $\alpha: a \rightarrow b$, $\beta: b \rightarrow c$, $\gamma: c \rightarrow d$ are maps, then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds;

(C₃) For each object $a \in C$ there is a map $\varepsilon_a: a \rightarrow a$, called the identity map of a such that for any $\alpha: b \rightarrow a$ and $\beta: a \rightarrow c$ we have $\alpha\varepsilon_a = \alpha$, $\varepsilon_a\beta = \beta$.

For the definitions of familiar concepts, such as monomorphism, epimorphism, equivalence, kernel, image, direct product, inverse limit, etc. we refer to the books [1], [3] and to the papers [4], [6], respectively. In this paper an epimorphism will always mean a normal epimorphism, and it will be supposed that the product of two normal epimorphisms is again a normal one. The subobject determined by the object a and the monomorphism α will be denoted by (a, α) . If the map $\xi: a \rightarrow b$ is an equivalence then we shall write $a \sim b(\xi, \xi^{-1})$ or only $a \sim b$, if there is no fear of ambiguity. The zero maps and identity maps will be denoted by ω and ε respectively, and the zero objects by 0 .

We say that a diagram consisting of rows and columns is *exact*, if its rows and columns are exact.

As it was done in [4] and [6], we shall suppose that the category C satisfies the following additional requirements:

(C₄) C possesses zero objects;

(C₅) Every map has a kernel;

(C₆) Every map has an image;

(C₇) An image of an ideal by an epimorphism is always an ideal;

(C₈) Every family of objects has a direct product and a free product;

(C₉) The class of all subobjects of any object is a set;

(C₁₀) For each object $a \in C$ the set of all ideals of a is a complete lattice;

(C₁₁) Every inverse system has an inverse limit.

We shall need the analogous statements of the Noetherian Isomorphism Theorems.

First Isomorphism Theorem. Let $(k, \kappa) \cong (d, \delta)$ be two ideals of an object a , and let

$$0 \rightarrow k \xrightarrow{\kappa} a \xrightarrow{\alpha} b \rightarrow 0$$

be an exact sequence. Denote the image of (d, δ) by the epimorphism α , by (m, χ) . Then there are such maps γ, β that

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & k & \rightarrow & d & \rightarrow & m \rightarrow 0 \\ & & & & \downarrow \delta & & \downarrow \chi \\ 0 & \rightarrow & k & \xrightarrow{\kappa} & a & \xrightarrow{\alpha} & b \rightarrow 0 \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & \rightarrow & c_1 \rightarrow c_2 \rightarrow 0 \\ & & & & \downarrow \gamma & & \downarrow \beta \\ & & & & 0 & & 0 \end{array}$$

is an exact commutative diagram.

For the proof we refer to [6] Theorem 2.1.

Second Isomorphism Theorem. Let $(k, \varkappa)$, (d_1, δ_1) and (d_2, δ_2) be ideals of an object $a \in C$ such that

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2),$$

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

are valid. If

$$0 \rightarrow k \rightarrow d_1 \rightarrow b_1 \rightarrow 0$$

and

$$0 \rightarrow d_2 \rightarrow a \rightarrow b_2 \rightarrow 0$$

are exact sequences, then the diagram

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \rightarrow & k & \rightarrow & d_1 & \rightarrow & b_1 \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & d_2 & \rightarrow & a & \rightarrow & b_2 \rightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

is exact and commutative.

The proof can be found in [4].

Combining the assertions of Theorem 2, 3 in [6] and Theorem 2, 5 of [4] we obtain

Proposition 1. Let

$$0 \rightarrow k \xrightarrow{\varkappa} a \rightarrow b \rightarrow 0$$

and

$$0 \rightarrow d_i \xrightarrow{\delta_i} a \rightarrow b_i \rightarrow 0 \quad (i = 1, 2)$$

be exact sequences. If

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2)$$

and

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold, then b is a direct product of b_1 and b_2 : $b = b_1 \times b_2$ (α_1, α_2 ; β_1, β_2) moreover, (b_i, β_i) ($i = 1, 2$) is the image of (d_i, δ_i) by α and

$$\text{Ker } \alpha_1 = (b_2, \beta_2), \quad \text{Ker } \alpha_2 = (b_1, \beta_1)$$

hold.

We can prove also the converse statement of this proposition.

Proposition 2. Let

$$0 \rightarrow k \xrightarrow{\varkappa} a \xrightarrow{\alpha} b_1 \times b_2 (\pi_i; \varrho_i) \quad (i = 1, 2)$$

be an exact sequence, and denote the complete counterimage of (b_i, ϱ_i) by (d_i, δ_i) ($i = 1, 2$). Then

$$(k, \varkappa) = (d_1, \delta_1) \cap (d_2, \delta_2)$$

and

$$(a, \varepsilon) = (d_1, \delta_1) \cup (d_2, \delta_2)$$

hold.

PROOF. The image of $(d_1, \delta_1) \cap (d_2, \delta_2)$ by α is $(b_1, \varrho_1) \cap (b_2, \varrho_2) = (0, \omega)$, so its complete counterimage by α is $\text{Ker } \alpha = (k, \varkappa)$.

Denote the union of (d_1, δ_1) and (d_2, δ_2) by (l, λ) . Now

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & l & \longrightarrow & b_0 & \longrightarrow 0 \\ & \downarrow & & \downarrow \varrho_0 & \\ 0 \rightarrow k \rightarrow a & \xrightarrow{\alpha} & b_1 \times b_2 & \rightarrow 0 \end{array}$$

is an exact commutative diagram such that $(b_0, \varrho_0) = (b_1, \varrho_1) \cup (b_2, \varrho_2)$ holds. Since by the Second Isomorphism Theorem

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & b_2 & \longrightarrow & b_2 & \longrightarrow & 0 \\ & \downarrow & \downarrow \varrho_2 & & \downarrow \varepsilon & & \\ 0 \rightarrow b_1 & \xrightarrow{\varrho_1} & b_1 \times b_2 & \xrightarrow{\pi_2} & b_2 & \rightarrow & 0 \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

is also an exact commutative diagram, so the complete counterimage of (b_2, ε) by π_2 is just (b_0, ϱ_0) . Since π_2 is an epimorphism, we get

$$(b_0, \varrho_0) = b_1 \times b_2 (\pi_2; \varrho).$$

Thus the assertion is proved.

We recall the definition of M -radical given in [6]. For this reason, consider a class M of objects belonging to C such that

- (i) If $a \in M$ and $a \sim b$ then $b \in M$;
- (ii) If $a, b \in M$ and $\alpha: a \rightarrow b$ is an epimorphism, then either α is a zero map, or $a \sim b(\alpha, \alpha_0^{-1})$.

The objects belonging to M will be called M -objects. We say that the ideal (d, δ) is an M -ideal of $a \in C$, if there is an exact sequence $0 \rightarrow d \xrightarrow{\delta} a \rightarrow b \rightarrow 0$ such that $b \in M$. The set M_a consisting of all M -ideals $(\neq (a, \varepsilon))$ of an object a is called the *structure M -space* of a .

The M -radical of an object $a \in C$ is the intersection of all M -ideals of a . If a has no M -ideals, then its M -radical is (a, ε) , and a is called an M -radical object.

A subset Δ of M_a is called an *independent M -system*, of a , if for any M -ideals $(d_1, \delta_1), \dots, (d_n, \delta_n) \in M_a$ and exact sequences

$$0 \rightarrow d_i \xrightarrow{\delta_i} a \rightarrow b_i \rightarrow 0 \quad (i = 1, \dots, n)$$

the sequence

$$0 \rightarrow \bigcap_{i=1}^n (d_i, \delta_i) \rightarrow a \rightarrow b_1 \times \dots \times b_n \rightarrow 0$$

is also exact. Every object has *maximal independent M-systems*, and the *M-radical* of an object is just the intersection of the *M-ideals* of any independent *M-system* of *a* (cf. [6]).

Any (not necessarily maximal) independent *M-system* Δ induces an inverse system Ω_Δ as follows (cf. [6]). Consider the set $\mathbf{F}_\Delta$ — a so-called *filter* — consisting of all finite intersections (c_i, γ_i) ($i \in I$) of *M-ideals* belonging to Δ . We regard I as a partially ordered set by the definition: $i \cong j$ if and only if $(c_i, \gamma_i) \cong (c_j, \gamma_j)$. For each $(c_i, \gamma_i) \in \mathbf{F}_\Delta$ let us consider an exact sequence

$$0 \rightarrow c_i \rightarrow a \xrightarrow{\alpha_i} b_i \rightarrow 0.$$

Using the First Isomorphism Theorem we get that for any ideals $(c_i, \gamma_i) \cong (c_j, \gamma_j)$ there is an epimorphism $\pi_j^i: b_i \rightarrow b_j$ such that

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & c_i & \rightarrow & c_j & \rightarrow & m_j^i \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & c_i & \rightarrow & a & \xrightarrow{\beta_i} & b_i \rightarrow 0 \\ & & \downarrow \beta_j & & \downarrow \pi_j^i & & \\ & & 0 & \rightarrow & b_j & \xrightarrow{\epsilon} & b_j \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

is an exact commutative diagram, further we have $\beta_i \pi_j^i \pi_k^i = \beta_i \pi_k^i$ for any $i \cong j \cong k \in I$. Since β_i is an epimorphism, there follows $\pi_j^i \pi_k^i = \pi_k^i$. Thus $\Omega_\Delta = [b_i, \pi_j^i]$ forms an inverse system, and there is a uniquely defined map, the so called *canonical map* of *a* belonging to Δ , $\beta: a \rightarrow \varprojlim \Omega_\Delta$ such that $\beta \pi_i = \beta_i$ ($i \in I$) holds where π_i means the projection $\pi_i: \varprojlim \Omega_\Delta \rightarrow a_i$.

It is obvious that if $\Delta = \{(d_k, \delta_k) | k \in K\}$ is an independent *M-system*, then the elements $k \in K$ are just the minimal elements of the index set I of $\mathbf{F}_\Delta$.

Theorem 1. *If*

$$0 \rightarrow d_k \rightarrow a \rightarrow a_k \rightarrow 0 \quad (k \in K)$$

are exact sequences, then $\varprojlim \Omega_\Delta$ is a direct product of the objects a_k ($k \in K$).

PROOF. According to the construction of $\Omega_\Delta = [b_i, \pi_j^i]$ any b_i is determined by the exact sequence

$$0 \rightarrow (c_i, \gamma_i) = \bigcap_{\text{finite}} (d_k, \delta_k) \rightarrow a \rightarrow b_i \rightarrow 0.$$

Since Δ is an independent *M-system*, b_i is a direct product $b_i = \prod_{\text{finite}} a_k(\tau_k^{(i)}, \vartheta_k^{(i)})$ of the objects a_k such that

$$0 \rightarrow d_k \rightarrow a \rightarrow a_k \rightarrow 0$$

is an exact sequence for all $k \in K$ occurring in the direct decomposition of b_i . Thus

by definition for the projections $\pi_i: \varprojlim \Omega_A \rightarrow b_i$ the relation $\pi_i \pi_j^i = \pi_j$ is valid for all $i \geq j \in I$, and for any system of maps $\chi_i: h \rightarrow b_i$ satisfying $\chi_i \pi_j^i = \chi_j$, there is a unique map $\chi: h \rightarrow \varprojlim \Omega_A$ such that $\chi \pi_i = \chi_i$ is valid for each $i \in I$.

We have to prove that $\varprojlim \Omega_A$ is a direct product of the objects a_k ($k \in K$). Obviously we have *)

$$\varrho_k^{(i)}: \varprojlim \Omega_A \rightarrow a_k$$

for $\varrho_k^{(i)} = \pi_i \tau_k^{(i)}$ ($k \in K \subseteq I$) and $\varrho_k^{(j)} = \varrho_k^{(i)}$ for $i \geq j$. Let h be an object and $\alpha_k: h \rightarrow a_k$ ($k \in K$) a system of maps. Now we can complete this system of maps by setting $\chi_i^{(k)} = \alpha_k \varrho_k^{(i)}: h \rightarrow b_i$. Clearly $\chi_i^{(k)} \pi_j^i = \chi_j^{(k)}$ is valid for all $i \geq j \in I$. Thus by the definition of the inverse limit there exists a unique map χ such that $\chi \pi_i = \chi_i^{(k)}$. So we get

$$\chi \varrho_k^{(i)} = \chi \pi_i \tau_k^{(i)} = \chi_i^{(k)} \tau_k^{(i)} = \alpha_k \varrho_k^{(i)} \tau_k^{(i)} = \alpha_k,$$

therefore $\varprojlim \Omega_A$ is indeed a direct product of the objects a_k ($k \in K$).

Again, let $\Delta = \{(d_k, \delta_k) | k \in K\}$ be an independent M -system of the object a .

Theorem 2. *If α denotes the canonical map $\alpha: a \rightarrow \varprojlim \Omega_A$ then $\text{Ker } \alpha = \bigcap_{k \in K} (d_k, \delta_k)$ holds.*

PROOF. Denote $\bigcap_{k \in K} (d_k, \delta_k)$ by (l, λ) . By Theorem 1 there is an equivalence

$$\zeta: \varprojlim \Omega_A \rightarrow \prod_{k \in K} a_k(\tau_k, \vartheta_k)$$

such that

$$0 \rightarrow d_k \rightarrow a \rightarrow a_k \rightarrow 0$$

is an exact sequence for each $k \in K$. This implies $\text{Ker } \alpha = \text{Ker } \beta$ ($\beta = \alpha \zeta$).

If $\beta_k = \beta \pi_k$, then we have $\lambda \beta_k = \omega$ for each $k \in K$. Thus ω is the uniquely defined map $\omega: l \rightarrow \prod_{k \in K} a_k(\tau_k, \vartheta_k)$ and this implies $\lambda \beta = \omega$.

Let $\gamma: c \rightarrow a$ be a map satisfying $\gamma \beta = \omega$. Now $\gamma \beta_k = \gamma \beta \pi_k = \omega$ holds for each $k \in K$, therefore there are maps $\gamma_k: c \rightarrow d_k$ ($k \in K$) such that $\gamma_k \delta_k = \gamma$. Because of $(l, \lambda) = \bigcap_{k \in K} (d_k, \delta_k)$ there exists a map $\gamma': c \rightarrow l$ such that $\gamma' \lambda = \gamma$ holds. Thus the assertion $\text{Ker } \beta = (l, \lambda)$ is proved.

As an immediate consequence of Theorem 2 we obtain

Proposition 3. *If Δ is a maximal independent M -system of a and α is the canonical map $\alpha: a \rightarrow \varprojlim \Omega_A$, then $\text{Ker } \alpha$ is just the M -radical of a .*

In [6] we have introduced the concept of M -compactness. Let us consider a maximal independent M -system Δ_0 of an object a . The object a is said to be M -compact, if the canonical map $\alpha: a \rightarrow \varprojlim \Omega_{\Delta_0}$ is an epimorphism.

* The maps having indices in brackets, essentially, do not depend on these indices.

Proposition 4. *If a is an M -compact object, then there is a one-to-one correspondence between the M -ideals of a and those of $\varprojlim \Omega_{\Delta_0}$.*

PROOF. Applying the First Isomorphism Theorem, we have the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & k & \rightarrow & d & \longrightarrow & m \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & k & \xrightarrow{\kappa} & a & \rightarrow & \varprojlim \Omega_{\Delta_0} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & c_1 & \longrightarrow & c_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

where (k, κ) means the M -radical of a , further (d, δ) and (m, χ) are M -ideals of a and $\varprojlim \Omega_{\Delta_0}$ respectively. Thus $(d, \delta) \leftrightarrow (m, \chi)$ induces a one-to-one correspondence between the M -ideals of a and $\varprojlim \Omega_{\Delta_0}$.

Let M_a denote the structure M -space of an object a . M_a induces a filter $\mathbf{F}_M$, and there belongs an inverse system Ω_M to $\mathbf{F}_M$. In [6] we have proved

Proposition 5. *If Δ_0 denotes a maximal independent M -system of a then the inverse system Ω_{Δ_0} is a cofinal subsystem of Ω_M , and so $\varprojlim \Omega_{\Delta_0} \sim \varprojlim \Omega_M$ holds.*

The class M satisfying the conditions (i) and (ii) is said to be a *modular class* of objects, if M satisfies also the following conditions (cf. [4], [6]):

(iii) If (p, π) is an ideal of a and $p \in M$, then there is a uniquely defined ideal $(m, \chi) \in M_a$ such that $(p, \pi) \cap (m, \chi) = (0, \omega)$;

(iv) If (l, λ) is an ideal of a , and (q, ϑ) is an M -ideal of l , then $(q, \vartheta\lambda)$ is an ideal of a .

A non-zero object a is called *simple* if its only ideals are $(0, \omega)$ and (a, ε) . Throughout this paper we suppose that M denotes a modular class of simple objects.

Proposition 6. (see [4], Proposition 4. 7). *Let (l, λ) be an ideal of an object $a \in C$. If (d, δ) is an M -ideal of a , then $(d, \delta) \cap (l, \lambda)$ is either an M -ideal of l or it is equal to (l, ε) . Conversely, any M -ideal (q, ϑ) of l can be represented as an intersection $(q, \vartheta\lambda) = (d, \delta) \cap (l, \lambda)$ where (d, δ) is an M -ideal of a . The correspondence $(q, \vartheta) \leftrightarrow (d, \delta)$ is one-to-one.*

As a consequence of Proposition 6 one can obtain

Proposition 7. (see [4] Theorem 4. 10) *If (l, λ) is an ideal of an object $a \in C$ and (c_a, γ_a) denotes the M -radical of a then $(c_l, \gamma_l\lambda) = (c_a, \gamma_a) \cap (l, \lambda)$ is valid where (c_l, γ_l) denotes the M -radical of l .*

§ 2. M -representable ideals

Following Suliński [4], we say that the ideal (l, λ) of an object $a \in C$ is *M -representable*, if (l, λ) can be represented as the intersection of all M -ideals containing it.

In this section we shall show a close connection between the M -compact objects and M -representable ideals. First we prove the following statement.

Proposition 8. *If $\Delta = \{(d_i, \delta_i) | i \in I\}$ is a not necessarily maximal independent M -system of an M -compact object a , and J, K are subsets of I such that $J \cup K = I$ and $J \cap K = \emptyset$, then*

$$\left(\bigcap_{j \in J} (d_j, \delta_j)\right) \cup \left(\bigcap_{k \in K} (d_k, \delta_k)\right) = (a, \varepsilon)$$

holds.

PROOF. Consider exact sequences

$$0 \rightarrow d_i \rightarrow a \rightarrow a_i \rightarrow 0 \quad (i \in I)$$

and denote $\bigcap_{i \in I} (d_i, \delta_i)$, $\bigcap_{j \in J} (d_j, \delta_j)$ and $\bigcap_{k \in K} (d_k, \delta_k)$ by (l, λ) , (l_J, λ_J) and (l_K, λ_K) , respectively. Since Δ is an independent M -system, so by Theorem 1 the sequences

$$0 \rightarrow l_J \rightarrow a \rightarrow \prod_{j \in J} a_j \rightarrow 0,$$

$$0 \rightarrow l_K \rightarrow a \rightarrow \prod_{k \in K} a_k \rightarrow 0,$$

$$0 \rightarrow l \rightarrow a \xrightarrow{\alpha} \prod_{j \in J} a_j \times \prod_{k \in K} a_k \rightarrow 0$$

are exact. Consider the complete counterimage of $\prod_{j \in J} a_j$ and $\prod_{k \in K} a_k$ by α , they are obviously (l_K, λ_K) and (l_J, λ_J) , respectively. Thus by Proposition 2

$$(l_J, \lambda_J) \cup (l_K, \lambda_K) = (a, \varepsilon)$$

holds, and the assertion is proved.

Theorem 3. *Let M be a modular class of simple objects. If (l, λ) is an M -representable ideal of an M -compact object a , then the object l is also M -compact.*

PROOF. Consider a maximal independent M -system $\Delta_1 = \{(d_j, \delta_j) | j \in J\}$ of M -ideals of a containing the ideal (l, λ) . By Zorn's Lemma there exists such an M -system.

We establish the relation $(l, \lambda) = \bigcap_{j \in J} (d_j, \delta_j)$. $(l, \lambda) \subseteq \bigcap_{j \in J} (d_j, \delta_j)$ is obviously true. To prove $(l, \lambda) \supseteq \bigcap_{j \in J} (d_j, \delta_j)$ let us consider the sequence

$$0 \rightarrow \bigcap_{j \in J} (d_j, \delta_j) \xrightarrow{\alpha} a \xrightarrow{\alpha} \varprojlim \Omega_{\Delta_1} \rightarrow 0,$$

where

$$\Delta_1 = \{(d_j, \delta_j) | j \in J\}.$$

According to Theorem 2 this sequence is exact. Consider a map $\gamma: c \rightarrow a$ such that $\gamma\alpha = \omega$. Since $\text{Ker } \alpha = \bigcap_{j \in J} (d_j, \delta_j)$ holds, there exists a map $\gamma': c \rightarrow \text{Ker } \alpha$ such that $\gamma'\alpha = \gamma$.

By the First Isomorphism Theorem we have for any M -ideal (d_j, δ_j) ($j \in J$) the following exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \rightarrow & d_j & \rightarrow & m_j \rightarrow 0 \\
 & & \downarrow & & \downarrow \alpha_j & & \\
 0 & \rightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \rightarrow & a & \xrightarrow{\alpha} & \varprojlim \Omega_{\Delta_1} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & \rightarrow & c_1 & \rightarrow & c_2 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

and it is easy to see that $\Delta' = \{(m_j, \chi_j) | j \in J\}$ forms a maximal independent M -system of $\varprojlim \Omega_{\Delta_1}$. If (d, δ) is an arbitrary M -ideal containing (l, λ) , and (m, χ) is its image by α , then with respect Proposition 5 to (m, χ) there is an intersection $\bigcap_{\text{finite}} (m_j, \chi_j)$ such that $\bigcap_{\text{finite}} (m_j, \chi_j) \subseteq (m, \chi)$. Hence for their complete counter-images $\bigcap_{\text{finite}} (d_j, \delta_j) \subseteq (d, \delta)$ holds. Thus there exists a map $\gamma_j: c \rightarrow d$ such that $\gamma_j \delta_j = \gamma$, moreover there exists a map $\gamma_0: c \rightarrow \bigcap_{(d, \delta) \subseteq (l, \lambda)} (d, \delta) = (l, \lambda)$ such that $\gamma_0 \lambda = \gamma$. Hence $(l, \lambda) = \text{Ker } \alpha$ is proved.

Complete the M -system Δ_1 to an independent M -system Δ of a . Thus we have $\Delta = \Delta_1 \cup \Delta_2$ where Δ_2 consists of certain M -ideals (d_k, δ_k) ($k \in K$) not containing (l, λ) . So $J \cap K = \emptyset$ is valid, further denote $J \cup K$ by I .

To prove the theorem we need

Proposition 9. Consider the ideals $(q_k, \vartheta_k \lambda) = (d_k, \delta_k) \cap (l, \lambda)$ ($k \in K$). The system $\Phi = \{(q_k, \vartheta_k) | k \in K\}$ is an independent M -system of the object l . If Δ is a maximal independent M -system, then so is Φ .

By Proposition 8 for any finite intersection $(f, \varphi) = \bigcap_{\text{finite}} (d_k, \delta_k)$ we have $(f, \varphi) \cup (l, \lambda) = (q, \varepsilon)$, and so, using the Second Isomorphism Theorem we obtain the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & (f, \varphi) \cap (l, \lambda) & \rightarrow & l & \rightarrow & b' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & f & \rightarrow & a & \rightarrow & \prod a_k \rightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

Thus Φ is an independent M -system of l .

Suppose Φ is not maximal, i.e. there exists an ideal (q, ϑ) of l , such that $\Phi' = \Phi \cup \{(q, \vartheta)\}$ is also independent. By Proposition 6

$$(q, \vartheta \lambda) = (d, \delta) \cap (l, \lambda)$$

holds for an M -ideal (d, δ) of a . Consider any finite intersection $\bigcap_{\text{finite}} (d_k, \delta_k)$. By Proposition 5 there are finite many ideals of $\mathcal{A}$ such that

$$\bigcap_{\text{finite}} (d_i, \delta_i) \subseteq \bigcap_{\text{finite}} (d_k, \delta_k) \cap (d, \delta) = (n, v)$$

holds. From Proposition 8 there follows

$$\left(\bigcap_{\text{finite}} (d_i, \delta_i) \right) \cup (l, \lambda) = (a, \varepsilon),$$

hence

$$(n, v) \cup (l, \lambda) = (a, \varepsilon)$$

is valid too.

The intersection $(n, v) \cap (l, \lambda) = (g, \psi)$ is obviously a finite intersection of ideals of Φ' and (q, ϑ) is one of the components of this intersection. Since Φ' is independent, there is an exact sequence

$$0 \rightarrow f \rightarrow l \rightarrow a' \times \prod_{\text{finite}} a_k \rightarrow 0$$

such that

$$0 \rightarrow q \rightarrow l \rightarrow a' \rightarrow 0$$

is an exact sequence. Using the Second Isomorphism Theorem we obtain the following exact commutative diagrams:

$$\begin{array}{ccccccc} & 0 & 0 & & 0 & & \\ & \downarrow & \downarrow & & \downarrow & & \\ 0 & \rightarrow & g & \rightarrow & l & \rightarrow & a' \times \prod_{\text{finite}} a_k \rightarrow 0 \\ & \downarrow & \downarrow & & \downarrow & & \\ 0 & \rightarrow & n & \rightarrow & a & \rightarrow & b \rightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

and

$$\begin{array}{ccccccc} & 0 & 0 & & 0 & & \\ & \downarrow & \downarrow & & \downarrow & & \\ 0 & \rightarrow & q & \rightarrow & l & \rightarrow & a' \rightarrow 0 \\ & \downarrow & \downarrow & & \downarrow & & \\ 0 & \rightarrow & d & \rightarrow & a & \rightarrow & a'' \rightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

Hence $\mathcal{A}_2 \cup \{(d, \delta)\}$ is an independent M -system such that (d, δ) does not contain (l, λ) . This contradicts the choice of $\mathcal{A}_2$, and thus the proposition results proved.

Now we continue the proof of Theorem 3. Assume that $\mathcal{A}$ is a maximal independent M -system of a . In view of Theorem 1 and Proposition 9 we obtain

$$(1) \quad \varprojlim \Omega_\Phi \sim \prod_{k \in K} a_k \sim \varprojlim \Omega_{\mathcal{A}_2}.$$

Let (l_0, λ_0) denote the M -radical of a , and denote $\bigcap_{k \in K} (d_k, \delta_k)$ by (l', λ') . Now $(l_0, \lambda_0) = (l, \lambda) \cap (l', \lambda')$ is valid, and Proposition 8 implies

$$(a, \varepsilon) = (l, \lambda) \cup (l', \lambda').$$

Taking into account the M -compactness of a , from Theorems 1 and 2 we infer by the Second Isomorphism Theorem that

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & & & \\
 & \downarrow & \downarrow & \downarrow & & & \\
 0 & \rightarrow & l_0 & \rightarrow & l & \rightarrow & b_0 \rightarrow 0 \\
 & \downarrow & \downarrow & \downarrow & & & \\
 0 & \rightarrow & l' & \rightarrow & a & \rightarrow & \prod_{k \in K} a_k \rightarrow 0 \\
 & & & & & & \downarrow \\
 & & & & & & 0
 \end{array}$$

is an exact commutative diagram. Hence by (1) $b_0 \sim \varinjlim \Omega_\phi$ is valid. Thus l is M -compact, and Theorem 3 is proved.

Corollary. *If a is an M -compact, M -semi-simple object, then every M -representable ideal of a is a direct factor of a .*

If (l, λ) is an M -representable ideal of a , then by Theorem 3 the object l is M -compact. Hence we have the exact commutative diagram

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \rightarrow & 0 & \rightarrow & l & \rightarrow & \prod a_k \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & 0 & \rightarrow & a & \rightarrow & \prod_{j \in J} a_j \times \prod_{k \in K} a_k \rightarrow 0
 \end{array}$$

and so (l, λ) is a direct factor of a .

The statement of this corollary is analogous to well-known facts in the category of modules (cf [5]).

Now we show by a simple example that an M -compact object a in which every M -representable ideal is a direct factor, need not be M -semi simple. For this sake, let a be a direct product of an M -radical object $l \neq 0$ and an object $p \in M$: $a = l \times p$ ($\pi_1, \pi_2; \varrho_1, \varrho_2$). It is obvious that $(l, \varrho_1) \neq (0, \omega)$ is the M -radical of a , and every M -representable ideal (namely only (l, ϱ_1)) is a direct factor of a .

The converse statement of Theorem 3 is also true.

Theorem 4. *Let M be a modular class of simple objects. If l and a are M -compact objects and (l, λ) is an ideal of a containing the M -radical of a , then (l, λ) is an M -representable ideal of a , moreover, (l, λ) is the intersection $\bigcap_{j \in J} (d_j, \delta_j)$ of M -ideals which form an independent M -system $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ of a .*

PROOF. Consider the set of all M -ideals of a containing (l, λ) , and choose a maximal independent M -system $\Delta_K = \{(d_k, \delta_k) | k \in K\}$ from these M -ideals. Complete Δ_K to a maximal independent M -system Δ of a . Obviously $\Delta = \Delta_K \cup \Delta_J$ holds where $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ denotes an independent M -system consisting of M -ideals not containing (l, λ) .

Let us denote the M -radical of a by (c, γ) . Taking into account Proposition 3 and Theorem 1, we obtain the exact sequence

$$0 \rightarrow c \rightarrow a \rightarrow \prod_{i \in I} a_i \rightarrow 0 \quad (I = K \cup J)$$

where a_i ($i \in I$) are M -objects such that

$$0 \rightarrow d_i \xrightarrow{\delta_i} a \rightarrow a_i \rightarrow 0$$

is an exact sequence, meanwhile (d_i, δ_i) ranges over the maximal independent M -system Δ .

Consider the set $\Phi = \{(q_k, \vartheta_k) | k \in K\}$ such that $(q_k, \vartheta_k \lambda) = (d_k, \delta_k) \cap (l, \lambda)$. According to Proposition 9 Φ is a maximal independent M -system of l .

Since l is an M -compact object so by Theorem 1 and Proposition 3 we get the exact sequence

$$0 \rightarrow c \xrightarrow{\gamma_1} l \rightarrow \prod_{k \in K} a_k \rightarrow 0$$

and here, according to Proposition 7 (c, γ_1) is the M -radical of l . The First Isomorphism Theorem yields the following exact commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & c & \rightarrow & l & \rightarrow & \prod_{k \in K} a_k \xrightarrow{\omega} 0 \\ & & \downarrow & & \downarrow \vartheta & & \\ 0 & \rightarrow & c & \rightarrow & a & \xrightarrow{\alpha} & \prod_{i \in I} a_i \rightarrow 0 \\ & & \downarrow \varphi & & \downarrow \pi & & \\ & & 0 & \rightarrow & b & \rightarrow & \prod_{j \in J} a_j \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Hence $\text{Ker } \varphi = (l, \lambda)$ holds.

According to Proposition 8 there follows from the Second Isomorphism Theorem that

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & c & \longrightarrow & \bigcap_{j \in J} (d_j, \delta_j) & \rightarrow & b' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \bigcap_{k \in K} (d_k, \delta_k) & \longrightarrow & a & \longrightarrow & \prod_{k \in K} a_k \rightarrow 0 \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

is an exact commutative diagram. Thus the image of $\bigcap_{j \in J} (d_j, \delta_j)$ by $\alpha\pi$ is $(0, \omega)$.

Hence $\bigcap_{j \in J} (d_j, \delta_j) \cong (l, \lambda)$ is valid. Thus $(l, \lambda) = \bigcap_{j \in J} (d_j, \delta_j)$ holds and (l, λ) is an M -representable ideal of a .

§ 3. M -closure operation

We can introduce a closure operation in the set of all subobjects of an object $a \in C$ in a natural way.

Again, let M denote a modular class of simple objects, and let (l, λ) be a subobject of the object a . Denote the intersection of all M -ideals of a containing (l, λ) by $(\bar{l}, \bar{\lambda})$. If the set of such M -ideals is empty, then we put $(\bar{l}, \bar{\lambda}) = (a, \varepsilon)$.

The ideal $(\bar{l}, \bar{\lambda})$ will be called the M -closure of the subobject (l, λ) .

In other words: consider the structure M -space of a , and denote the set of all intersections of M -ideals by C . C is a closure-system, and the closure operation belonging to C is just the M -closure operation. The ideals with $(l, \lambda) = (\bar{l}, \bar{\lambda})$ (i.e. the M -representable ideals) will be called M -closed ideals.

Now the question arises: *does the M -closure operation define a topology on a ?* (i.e.: Is the union of finitely many M -closed ideals again closed?) Concerning this problem we establish the following result.

Theorem 5. *If a is an M -compact object, then the M -closure operation defines a topology on a .*

PROOF. It is sufficient to prove that the union of any two M -closed ideals (l_1, λ_1) , (l_2, λ_2) is again M -closed.

Consider the set of all M -ideals containing both of (l_1, λ_1) and (l_2, λ_2) , and choose a maximal independent M -system $\Delta_G = \{(d_g, \delta_g) | g \in G\}$ of such ideals. Complete Δ_G to maximal independent M -systems $\Delta_J = \{(d_j, \delta_j) | j \in J\}$ and $\Delta_K = \{(d_k, \delta_k) | k \in K\}$ of M -ideals containing (l_1, λ_1) and (l_2, λ_2) , respectively.

It is obvious that $J \cap K = G$ is valid, moreover $\{(d_i, \delta_i) | i \in (J \cup K) \setminus G\}$ forms an independent M -system of a .

Since by Theorem 3 l_1 and l_2 are M -compact objects, taking into account Theorem 4 we get $(l_1, \lambda_1) = \bigcap_{j \in J} (d_j, \delta_j)$ and $(l_2, \lambda_2) = \bigcap_{k \in K} (d_k, \delta_k)$.

Consider the ideal

$$(m, \chi) = \begin{cases} \bigcap_{g \in G} (d_g, \delta_g) & \text{if } G \neq \emptyset \\ (a, \varepsilon) & \text{if } G = \emptyset \end{cases}$$

and the ideals (d_i, δ_i) ($i \in (J \cup K) \setminus G$). To each of them there exists an M -ideal (m_i, χ_i) of m such that

$$(m_i, \chi_i) = (d_i, \delta_i) \cap (m, \chi)$$

holds. From Proposition 9 it follows that

$$\Phi = \{(m_i, \chi_i) | i \in (J \cup K) \setminus G\}$$

is an independent M -system of m . By definition we have

$$(l_1, \lambda_1) = \bigcap_{j \in J} (d_j, \delta_j) = \bigcap_{j \in J \setminus G} (d_j, \delta_j) \cap (m, \chi) = \bigcap_{j \in J \setminus G} (m_j, \chi_j)$$

and

$$(l_2, \lambda_2) = \bigcap_{k \in K} (d_k, \delta_k) = \bigcap_{k \in K \setminus G} (d_k, \delta_k) \cap (m, \chi) = \bigcap_{k \in K \setminus G} (m_k, \chi_k).$$

Since $J \setminus G$ and $K \setminus G$ are disjoint sets, Proposition 8 implies

$$\left(\bigcap_{j \in J \setminus G} (m_j, \chi_j) \right) \cup \left(\bigcap_{k \in K \setminus G} (m_k, \chi_k) \right) = (m, \varepsilon).$$

Thus

$$(l_1, \lambda_1) \cup (l_2, \lambda_2) = \begin{cases} \bigcap_{g \in G} (d_g, \delta_g) & \text{if } G \neq \emptyset \\ (a, \varepsilon) & \text{if } G = \emptyset \end{cases}$$

is valid and the theorem is proved.

In [2] LEPTIN has proved (among other results) that the sum of two closed submodules of a linearly compact module is again closed, and so linearly compact. The same holds for so-called 'in a narrow sense linearly compact' modules, i.e. for modules which are inverse limits of modules satisfying the descending chain condition. As it has been remarked in [6], the concept of M -compactness is a generalization of this concept. So Theorem 5 can be regarded as a category-theoretical generalization of LEPTIN's result.

The subobject (l, λ) of an object $a \in C$ will be called *dense* in a , if its M -closure is (a, ε) .

By virtue of this definition, it is clear that a dense subobject (l, λ) cannot be contained in any M -ideal of a . Moreover, according to Proposition 6 there is a one-to-one correspondence between the M -ideals of a dense ideal (l, λ) and those of a .

If $\bar{a}$ is an M -compact object such that (a, α) is a dense subobject of $\bar{a}$, then $\bar{a}$ will be called an *M -compactification* of a .

If $\bar{a}_1$ and $\bar{a}_2$ are M -compactifications of a , then by the remark made above and in view of the Second Isomorphism Theorem, it is clear that the inverse systems $\Omega_a, \Omega_{\bar{a}_1}$ and $\Omega_{\bar{a}_2}$ induced by the structure M -spaces of $a, \bar{a}_1$ and $\bar{a}_2$, respectively, are equivalent. However, $\bar{a}_1$ and $\bar{a}_2$ need not be equivalent objects. For instance, consider an M -radical object $l \neq 0$ and a direct product $\bar{a}_1 = \prod_{i=1}^{\infty} a_i(\pi_i, \varrho_i)$ of M -objects $a_1, a_2, \dots$. If a is the discrete direct product $\bigcup_{i=1}^{\infty} (a_i, \varrho_i)$, then both $\bar{a}_1$ and $\bar{a}_2 = \bar{a}_1 \times l$ are M -compactifications of a but $\bar{a}_1$ and $\bar{a}_2$ are not equivalent.

Problem. *Has any object an M -compactification?* If no, then give a necessary and sufficient condition for the existence of M -compactifications of an object.

Now we prove the existence of an M -compactification of M -semi-simple objects (they are objects whose M -radical is a zero object).

Theorem 6. *Every M -semi-simple object has an M -compactification, moreover, every M -semi-simple object is a dense subobject of a direct product of M -objects.*

PROOF. By Proposition 3 and Theorem 1 $a \in C$ can be embedded by a monomorphism α in a direct product of M -simple objects which is M -compact and M -semi-simple. If (d, δ) is the M -closure of (a, α) , then according to Theorem 3 d is an M -compact object, moreover, by Corollary 4, 12 of SULIŃSKI [4] also d is M -semi-simple. Thus d is a direct product of M -objects. By Proposition 6 a can be embedded as a dense subobject in d .

Theorem 6 generalizes the statement of Satz 15 in LEPTIN [2]. This theorem asserts that any semi-simple ring can be embedded as a dense subring in a direct product of linearly compact (topologically) simple rings.

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