

On the iteration of the divisor-function

By I. KÁTAI (Budapest)

1. Let $d(n)$ denote the number of divisors of n , and let

$$(1.1) \quad d_r(n) = d_{r-1}(d(n)), \quad r = 2, 3, \dots; \quad d_1(n) = d(n).$$

Let further

$$(1.2) \quad D_r(x) = \sum_{n \leq x} d_r(n).$$

It is well-known, that

$$D_1(x) = (1 + o(1))x \log x, \quad \text{as } x \rightarrow \infty.$$

R. BELLMAN and H. SHAPIRO [1] called the attention to the investigation of $D_r(x)$ for $r \geq 2$. It was conjectured by them, that

$$(1.3) \quad D_r(x) = c_r(1 + o(1))x \log_r x \quad \text{as } x \rightarrow \infty$$

for every $r \geq 1$, where $\log_r x$ denotes the r -fold iterated logarithmus of x , i.e.

$$(1.4) \quad \log_r x = \log(\log_{r-1} x), \quad \log_1 x = \log x; \quad r = 2, 3, \dots$$

It was remarked in a footnote that P. ERDŐS proved the relation (1.3) for $r=2$.

The aim of this paper is to prove the relation (1.3) for $r=3$.

The proof of (1.3) seems to be very difficult for $r \geq 4$.

Let

$$(1.5) \quad \bar{D}_r(x) = \sum_{p \leq x} d_r(p-1),$$

where in the sum p runs over the prime numbers.

We shall prove that

$$(1.6) \quad \bar{D}_r(x) = c'_r(1 + o(1)) \frac{x}{\log x} \log_r x$$

in the case $r=2$.

We remark that the validity of (1.6) for $r=1$ (which is a more difficult problem) was proved by YU. V. LINNIK [2].

It seems to be difficult to verify the relation (1.6) for $r \geq 3$. We formulate now our assertions.

Theorem 1.

$$(1.7) \quad D_2(x) = (1 + o(1))c_1 x \log_2 x,$$

$$(1.8) \quad \bar{D}_2(x) = (1 + o(1))c_2 \frac{x}{\log x} \log_2 x,$$

as x tends to infinity, where c_1, c_2 denote suitable positive constants.

Theorem 2.

$$(1.9) \quad D_3(x) = (1 + o(1))c_3 x \log_3 x$$

as x tends to infinity, where c_3 denotes a suitable positive constant.*

For the proof of our assertions we need a theorem due to ERDŐS [3] which we state as

Lemma 1. Denote $P_k(x)$ the number of square-free integers $\leq x$ having exactly k prime factors. Then

$$(1.10) \quad P_k(x) = (1 + o(1)) \frac{6}{\pi^2} \frac{x}{\log x} \frac{(\log_2 x)^{k-1}}{(k-1)!}$$

uniformly for every k in the interval $J_x(c)$ defined by

$$(1.11) \quad \log_2 x - c (\log_2 x)^{1/2} \leq k \leq \log_2 x + c (\log_2 x)^{1/2},$$

where c is an arbitrary constant.

Using contour-integration and a theorem of Esseen we could improve this lemma (for this see Kubilius's book [4], Ch. 9.).

This lemma suggests me the following theorem concerning the asymptotic behavior of the iteration of the indicator function of the square-free numbers.

Let $\mu_1(n)$ be an arithmetical function defined by

$$\mu_1(n) = \begin{cases} 1, & \text{if } n \text{ is square-free, or } 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $U(n)$ denote as usual the number of different prime divisors of n , i.e. let $U(n)=r$ for $n=p_1^{z_1} \dots p_r^{z_r}$. Let further

$$\mu_2(n) = \mu_1(U(n))\mu_1(n),$$

$$U_k(n) = U(U_{k-1}(n)), \quad U_1(n) = U(n),$$

$$\mu_k(n) = \mu_1(U_{k-1}(n))\mu_1(U_{k-2}(n)) \dots \mu_1(n) = \mu_1(U_{k-1}(n))\mu_{k-1}(n); \quad k=3, 4, \dots$$

In the case $\mu_k(n)=1$ we call n a k -fold square-free number. We call a natural number n total-square free, if it is k -fold square-free for every k satisfying $U_k(n) \geq 1$. Let $\mu^{(T)}(n)$ denote the indicator of the set of total-square-free numbers, and let

$$M_l(x) = \sum_{n \leq x} \mu_l(n), \quad l = 1, 2, \dots;$$

$$M^{(T)}(x) = \sum_{n \leq x} \mu^{(T)}(n).$$

* P. Erdős and I proved, that $D_4(x) \sim c_4 x \log_4 x$

From Lemma 1 we can deduce very simply that

$$(1.12) \quad M_l(x) = (1 + o(1)) \left(\frac{6}{\pi^2} \right)^l x \quad (x \rightarrow \infty)$$

in the case $l=2$.

It seems probable, that (1.12) holds for every l , however we are unable to prove this for $l \geq 3$.

Let $k(x)$ be an integer valued function defined on the interval $e \leq x < \infty$ as follows:

$$1 \leq \log_k x < e, \quad k = k(x).$$

Perhaps the following relation holds:

$$\log \frac{M^T(x)}{x} = (1 + o(1)) \frac{k(x)}{2} \log \frac{6}{\pi^2}.$$

In the following $c, c_1, c_2, \dots$ denote positive constants, not the same at every places.

2. The proof of Theorem 1.

Let $\mathfrak{A}$ denote the set of those natural numbers n , which have every prime divisors at least on the second power, and let

$$(2.1) \quad \vartheta(x) = \sum_{\substack{n \leq x \\ n \in \mathfrak{A}}} 1.$$

From the Perron-formula we have

$$\vartheta(x) = \frac{1}{2\pi i} \int_{(\frac{1}{2} + \varepsilon)} \frac{x^s}{s} \prod_p \left\{ 1 + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots \right\} ds,$$

whence applying the contour-integral-technique we can deduce the following

Lemma 2.

$$(2.2) \quad \vartheta(x) = cx^{1/2} + O(x^{\vartheta}),$$

where $\vartheta < \frac{1}{2}$ is a suitable positive constant. Every natural number n can be represented uniquely in the form

$$(2.3) \quad n = Km; \quad K \in \mathfrak{A}, \quad (m, K) = 1, \quad \mu(m) \neq 0.$$

Let

$$(2.4) \quad d(K) = k;$$

$$(2.5) \quad k = 2^{\beta} k_1,$$

where k_1 is an odd integer. Then we have

$$(2.6) \quad d(d(n)) = d\left(\frac{k}{2^{\beta}}\right)(\beta + 1 + U(m)) = d(k) + d(k_1)U(m).$$

Let us now introduce the following notations.

$$(2.7) \quad T(y, K) = \sum_{\substack{m \leq y \\ (m, K) = 1}} |\mu(m)| U(m),$$

$$(2.8) \quad Z(y, K) = \sum_{\substack{m \leq y \\ (m, K) = 1}} |\mu(m)|.$$

The following inequalities are evident.

$$(2.9) \quad Z(y, K) \ll y,$$

$$(2.10) \quad T(y, K) \ll y \log_2 y.$$

Lemma 3.

$$(2.11) \quad Z(y, K) = A_K y + O(Ky^{1/2}),$$

$$(2.12) \quad T(y, K) = A_K y \log_2 y + O(y \log_3 y),$$

where

$$A_K = \frac{\varphi(K)}{\zeta(2)K} \prod_{p|K} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

PROOF. Using that

$$\sum_{\substack{(n, K) = 1 \\ n \leq z}} 1 = \frac{\varphi(K)}{K} z + O(K)$$

and

$$\sum_{\substack{\delta^2 \leq y \\ (\delta, K) = 1}} \frac{\mu(\delta)}{\delta^2} = \frac{1}{\zeta(2)} \prod_{p|K} \left(1 - \frac{1}{p^2}\right)^{-1} + O(y^{-1/2}),$$

we have

$$Z(y, K) = \sum_{\substack{\delta^2 \leq y \\ (\delta, K) = 1}} \mu(\delta) \sum_{\substack{u \leq y/\delta^2 \\ (u, K) = 1}} 1 = \sum_{\substack{\delta^2 \leq y \\ (\delta, K) = 1}} \mu(\delta) \left\{ \frac{y}{\delta^2} \frac{\varphi(K)}{K} + O(K) \right\} = A_K y + O(Ky^{1/2}),$$

and hence it follows (2.11).

For the proof of (2.12) let us put

$$\begin{aligned} T(y, K) &= \sum_{\substack{p \leq y \\ (p, K) = 1}} \sum_{\substack{np \leq y \\ (n, K) = 1}} |\mu(np)| = \sum_{\substack{p \leq y \\ (p, K) = 1}} \sum_{\substack{(n, pK) = 1 \\ n \leq y/p}} |\mu(n)| = \sum_{\substack{(p, K) = 1 \\ p \leq y}} Z\left(\frac{y}{p}, pK\right) = \\ &= \sum_{p \leq y^{0,1}} + \sum_{p > y^{0,1}} = \Sigma_1 + \Sigma_2. \end{aligned}$$

From (2.9),

$$\Sigma_2 \ll y.$$

Now using (2.11), we have

$$\Sigma_1 = A_K y \sum_{\substack{p < y^{0,1} \\ (p, K) = 1}} \frac{1}{p} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^2}\right)^{-1} + O(Ky^{0,65}).$$

Further the sum on the right hand side equals to

$$\sum_{\substack{p < y^{0,1} \\ (p,K)=1}} \frac{1}{p} + O(1) = \log_2 y - \sum_{\substack{p < y^{0,1} \\ p|K}} \frac{1}{p} + O(1).$$

Now we shall prove, that

$$\sum_{p|K} \frac{1}{p} \ll \log_3 K$$

and hence (2. 12) follows.

It is known that

$$\sigma(n) \ll n \log_2 n,$$

where $\sigma(n)$ denotes the sum of the divisors of n . Hence we obtain that

$$\sum_{p|K} \frac{1}{p} \ll \log \prod_{p|K} \left(1 + \frac{1}{p}\right) \ll \log \frac{\sigma(K)}{K} \ll \log_3 K.$$

Lemma 4. For $y \geq 1$

$$(2.13) \quad \sum_{\substack{K \leq y \\ K \in \mathfrak{H}}} \frac{d(k_1)}{K} \ll y^{-1/2+\varepsilon}$$

holds where $\varepsilon > 0$ is an $\varepsilon > 0$ arbitrary constant.

PROOF. Using the inequality $dd(n) < d(n) \ll n^\varepsilon$ we have

$$\sum_{M \leq K \leq 2M} d(d(K)) \ll 9(2M)M^\varepsilon \ll M^{1/2+\varepsilon}.$$

Hence

$$\sum_{K \leq y} \frac{d(k_1)}{K} = \sum_{v=0}^{\infty} \frac{1}{2^v y} \sum_{2^v y \leq K \leq 2^{v+1} y} d(k_1) \ll \sum_{v=0}^{\infty} \frac{1}{2^v y} (2^v y)^{1/2+\varepsilon} \ll y^{-1/2+\varepsilon}.$$

The proof of (1. 7) is straightforward. From (2. 3) it follows, that

$$D_2(x) = \sum_{\substack{K \leq x \\ K \in \mathfrak{H}}} d(k) Z\left(\frac{x}{K}, K\right) + \sum_{\substack{K \leq x \\ K \in \mathfrak{H}}} d(k_1) T\left(\frac{x}{K}, K\right) = \Sigma_3 + \Sigma_4.$$

Firstly we have evidently, that

$$\Sigma_3 \ll x \sum_{K \in \mathfrak{H}} \frac{d(k)}{K} \ll x.$$

Let now y be any number in the interval $1 \leq y \leq x$, and let

$$\Sigma_4 = \sum_{K \leq y} + \sum_{K > y} = \Sigma_5 + \Sigma_6.$$

Using (2. 10) and Lemma 4 we have

$$\Sigma_6 \ll x \log_2 x \sum_{K > y} \frac{d(k_1)}{K} \ll x \log_2 x \cdot y^{-1/2+\varepsilon} \ll x,$$

if

$$(2.14) \quad y \cong \log x.$$

Applying now (2.12) we obtain

$$\Sigma_5 = A_K x \log_2 x \sum_{\substack{K \in \mathfrak{H} \\ K < y}} \frac{d(k_1)}{K} + O(x \log_3 x) = c_1 x \log_2 x + O(x \log_3 x),$$

where

$$c_1 = A_K \sum_{K \in \mathfrak{H}} \frac{d(k_1)}{K}.$$

Hence the relation (1.7) follows immediately.

For the proof of (1.8) we need the following Turán type inequality

Lemma 5.

$$(2.15) \quad \sum_{\substack{p \equiv 1 \pmod{Q} \\ p \leq x}} \left(U\left(\frac{p-1}{Q}\right) - \log_2 x \right)^2 \ll \frac{x}{\varphi(Q) \log x} \log_2 x \cdot \log_3 x,$$

uniformly for $Q \leq (\log x)^{10}$.

For the proof see e.g. H. HALBERSTAM [5] (p. 24). Using the new result of Bombieri concerning the distribution of prime numbers in arithmetical progressions we could give a better estimation for (2.15).

Using the notations (2.3)–(2.6) we have

$$\bar{D}_2(x) = \sum_{K < x} d(k) \sum_{\substack{p-1=Km \\ p \leq x \\ (K,m)=1}} |\mu(m)| + \sum_{K < x} d(k_1) \sum_{\substack{(m,K)=1 \\ p-1=Km}} U(m) |\mu(m)| = \Sigma_7 + \Sigma_8.$$

Using Lemma 4 and the well known Brun–Titchmarsh inequality stating that

$$\pi(x, K, l) \ll \frac{x}{\varphi(K) \log x}$$

for $K < x^{1/2}$, we have

$$\Sigma_7 \ll \frac{x}{\log x} \sum_{K < x^{1/2}} \frac{d(k)}{\varphi(K)} + x \sum_{x^{1/2} \leq K < x} \frac{d(k)}{K} \ll \frac{x}{\log x}.$$

Let us now choose $y = (\log x)^{10}$, say, and let

$$\Sigma_8 = \sum_{K \leq y} + \sum_{K > y} = \Sigma_9 + \Sigma_{10}.$$

By Lemma 4 and the inequality

$$\sum_{m \leq x} U(m) \ll x \log_2 x$$

we obtain

$$\Sigma_{10} \ll \sum_{K > y} d(k_1) \sum_{m \leq \frac{x}{K}} U(m) \ll x \log_2 x \sum_{K > y} \frac{d(k')}{K} \ll x / \log^2 x.$$

Summarizing our results we have

$$\bar{D}_2(x) = \Sigma_9 + O\left(\frac{x}{\log x}\right).$$

Let now

$$\Sigma_9 = \log_2 x \sum_{K < y} d(k_1) \sum_{\substack{p-1=Km \\ (K,m)=1 \\ m \leq \frac{x}{K}}} |\mu(m)| + \Sigma_{12} = \log_2 x \Sigma_{11} + \Sigma_{12},$$

where

$$\Sigma_{12} = \sum_{K < y} d(k_1) \sum_{\substack{p-1=Km \\ (K,m)=1 \\ m \leq \frac{x}{K}}} (U(m) - \log_2 x) |\mu(m)|.$$

Using Lemma 5 and the Schwarz-inequality we have

$$\Sigma_{12} \ll \sum_{K < y} d(k_1) \pi^{1/2}(x, K, 1) \left\{ \sum_{\substack{p-1=Km \\ p \leq x}} (U(m) - \log_2 x)^2 \right\}^{1/2} \ll \frac{x}{\log x} (\log_2 x \cdot \log_3 x)^{1/2}.$$

Introducing the notation

$$T_K = \sum_{\substack{p-1=Km \\ (K,m)=1 \\ p \leq x}} |\mu(m)|$$

we have

$$\Sigma_9 = \sum_{K < y} d(k_1) T_K.$$

Now we shall give an asymptotic formula for T_K .

$$\begin{aligned} T_K &= \sum_{\substack{p-1=Km \\ (K,m)=1 \\ p \leq x}} \sum_{\delta^2/m} \mu(\delta) = \sum_{\substack{\delta^2 \leq x \\ (\delta, K)=1}} \mu(\delta) \sum_{d|K} \mu(d) \sum_{\substack{p \equiv 1 \pmod{K\delta^2 d} \\ p \leq x}} 1 = \\ &= \sum_{\substack{\delta^2 \leq x \\ (\delta, K)=1}} \mu(\delta) \sum_{d|K} \mu(d) \pi(x, d\delta^2, 1). \end{aligned}$$

Hence one can deduce easily the relation

$$T_K = \prod_{p|K} \left(1 - \frac{1}{p(p-1)}\right) \frac{x}{K \log x} \left(1 + O\left(\frac{1}{\log x}\right)\right),$$

uniformly for $K \leq y$, using the quoted theorem of Brun—Titchmarsh and the theorem of Siegel—Walfisz. Hence

$$\Sigma_9 = c_2 \frac{x \log_2 x}{\log x} + O\left(\frac{x}{\log x} (\log_2 x \cdot \log_3 x)^{1/2}\right)$$

follows, where

$$c_2 = \sum_{\substack{K=1 \\ K \in \mathfrak{H}}}^{\infty} \frac{d(k_1)}{K} \prod_{p|K} \left(1 - \frac{1}{p(p-1)}\right).$$

Therefore

$$D_2(x) = c_2 \frac{x \log_2 x}{\log x} \left(1 + O \left(\left(\frac{\log_3 x}{\log_2 x} \right)^{1/2} \right) \right)$$

and so the relation (1.8) holds.

3. The proof of Theorem 2.

Lemma 6. Let A be a natural number with a canonical representation

$$A = \prod p^{l_p}$$

and let

$$(3.1) \quad \Delta_A(x) = \sum_{n \leq x} d(An).$$

Then

$$(3.2) \quad \Delta_A(x) = c_0(A)x \log x + c_1(A)x + O(x^{1/3})$$

uniformly in $A \ll (\log x)^{10}$, where

$$(3.3) \quad c_0(A) = d(A) \prod_{p|A} \left(1 - \frac{l_p/(l_p+1)}{p} \right),$$

and

$$(3.4) \quad c_1(A) = c_0(A) \sum_{p|A} \frac{l_p}{l_p+1} \frac{\log p}{p - \frac{l_p}{l_p+1}} + c_1(1)c_0(A).$$

Furthermore we have

$$(3.5) \quad c_0(A) \ll d(A),$$

$$(3.6) \quad c_1(A) \ll d(A) \log^2 A.$$

It can be easily verified, that the function

$$f(s) = \sum_{n=1}^{\infty} \frac{d(An)}{n^s}$$

can be written in the form

$$f(s) = \zeta^2(s) \prod_{p|A} \left(l_p + 1 - \frac{l_p}{p^s} \right),$$

and that

$$f(s) = \frac{c_0(A)}{(s-1)^2} + \frac{c_1(A)}{s-1} + \dots$$

in some neighborhood of $s=1$.

Using the same analytical method which was elaborated for the Dirichlet divisor problem (see Titchmarsh [6], Ch. XII) we obtain (3.2).

The relations (3.5)—(3.6) from (3.3)—(3.4) immediately follow.

Let $P(x, K, r)$ denote the number of integers m for which

$$m \leq x, \quad |\mu(m)| = 1, \quad U(m) = r, \quad (K, m) = 1$$

are satisfied.

Let $V(n)$ denote the total number of the primedivisors of n , i.e. let $V(n) = \alpha_1 + \dots + \alpha_s$ for $n = p_1^{\alpha_1} \dots p_s^{\alpha_s}$.

Let further $\lambda(n) = (-1)^{V(n)}$, the so called Liouville-function.

Let $\mathcal{B}_K$ denote the set of all integers n , each of whose primedivisor is a divisor of K , i.e. for $K = p_1^{\alpha_1} \dots p_s^{\alpha_s}$ let

$$\mathcal{B}_K = \{n; n = p_1^{\beta_1} \dots p_r^{\beta_r}, \beta_i = 0, 1, \dots, i = 1, \dots, r\}.$$

For the sake of brevity let us denote

$$P(x, r) = P(x, 1, r).$$

The following relation holds.

Lemma 7.

$$(3.7) \quad P(x, K, r) = \sum_{\substack{v \leq x \\ v \in \mathcal{B}_K}} P\left(\frac{x}{v}, r - V(v)\right) \lambda(v).$$

For the proof of (3.7) we start from the identity

$$\begin{aligned} f_K(s) &\stackrel{\text{def}}{=} \sum_{\substack{(m, K)=1}} \frac{z^{U(m)} |\mu(m)|}{m^s} = \prod_{p \nmid K} \left(1 + \frac{z}{p^s}\right) = f_1(s) \sum_{v \in \mathcal{B}_K} \frac{\lambda(v) z^{V(v)}}{v^s} = \\ &= \sum \frac{z^{U(m)} |\mu(m)|}{m^s} \sum_{v \in \mathcal{B}_K} \frac{\lambda(v) z^{V(v)}}{v^s}. \end{aligned}$$

Comparing the coefficients on the left and right hand sides we have

$$\sum_{\substack{U(m)=r \\ (m, K)=1}} \frac{|\mu(m)|}{m^s} = \sum_{v \in \mathcal{B}_K} \frac{\lambda(v)}{v^s} \left\{ \sum_{U(n)=r-V(v)} \frac{|\mu(n)|}{n^s} \right\},$$

from which (3.7) immediately follows.

Lemma 8. Let $I_x(c)$ denote the interval

$$(3.8) \quad \log_2 x - c (\log_2 x)^{1/2} \leq r \leq \log_2 x + c (\log_2 x)^{1/2},$$

where c is an arbitrary positive constant. Then

$$(3.9) \quad P(x, K, r) = \frac{6}{\pi^2} (1 + o(1)) \prod_{p \mid K} \left(1 + \frac{1}{p}\right)^{-1} \frac{x}{\log x} \frac{(\log_2 x)^{r-1}}{(r-1)!}$$

uniformly for $r \in I_x(c)$ and $K \ll (\log_2 x)^4$.

Hence it follows, that the relation (3.9) holds uniformly for a suitable sequence of the c_x tending to infinity as $x \rightarrow \infty$.

PROOF. We need the following estimation:

$$\sum_{\substack{v \in \mathcal{B}_K \\ v > \Delta}} \frac{1}{v} < \sum_{v \in \mathcal{B}_K} \frac{1}{v} \left(\frac{v}{\Delta}\right)^{1/2} \leq \frac{1}{\Delta^{1/2}} \prod_{p \mid K} \left(1 + \frac{1}{p^{1/2}}\right)^{-1} \ll \frac{d(K)}{\Delta^{1/2}}.$$

From Lemma 7 we have

$$P(x, K, r) = \sum_{\substack{v \leq \Delta \\ v \in \mathcal{B}_K}} \lambda(v) P\left(\frac{x}{v}, r - V(v)\right) + O\left(\sum_{\substack{v > \Delta \\ v \in \mathcal{B}_K}} P\left(\frac{x}{v}, r - V(v)\right)\right).$$

Choosing $\Delta = (\log_2 x)^6$ the inequality

$$\sum_{\substack{v > \Delta \\ v \in \mathcal{B}_K}} P\left(\frac{x}{v}, r - V(v)\right) \ll x \sum_{\substack{v > \Delta \\ v \in \mathcal{B}_K}} \frac{1}{v} \ll x \frac{d(K)}{\Delta^{1/2}} \ll \frac{x}{(\log \log x)^2}$$

holds for the remainder term.

Let us suppose, that $r \in I_x(c)$. Then $r - V(v) \in I_x(2c)$, if $v \leq \Delta$. Applying Lemma 1 we have

$$P(x, K, r) = \frac{6}{\pi^2} (1 + O(1)) x \sum_{\substack{v < \Delta \\ v \in \mathcal{B}_K}} \frac{\lambda(v)}{v} \frac{\left(\log_2 \frac{x}{v}\right)^{r-1-V(v)}}{\log \frac{x}{v} \cdot (r-1-V(v))!} + O\left(\frac{x}{(\log_2 x)^2}\right).$$

Using that $V(v) \ll \log \Delta \ll \log_3 x$, we have

$$\frac{\left(\log_2 \frac{x}{v}\right)^{r-1-V(v)}}{\log \frac{x}{v} \cdot (r-1-V(v))!} = (1 + O(1)) \frac{(\log_2 x)^{r-1}}{\log x \cdot (r-1)!}$$

uniformly for $r \in I_x(c)$, $v \leq \Delta$. Taking into account, that

$$\sum_{\substack{v < \Delta \\ v \in \mathcal{B}_K}} \frac{\lambda(v)}{v} = \prod_{p|K} \left(1 + \frac{1}{p}\right)^{-1} + O\left(\frac{d(K)}{\Delta^{1/2}}\right),$$

and that

$$\frac{x}{(\log_2 x)^2} = o\left(\frac{x}{\log x} \frac{(\log_2 x)^{r-1}}{(r-1)!} \prod_{p|K} \left(1 + \frac{1}{p}\right)^{-1}\right)$$

for $r \in I_x(c)$, hence (3.9) follows.

We are now in the position to prove Theorem 2. We have, that

$$D_3(x) = \sum_{\substack{K \leq x \\ K \in \mathfrak{A}}} \sum_{\substack{n=Km \\ n \leq x}} d_3(n) = \sum_{K \leq y} + \sum_{K > y} = \Sigma_1 + \Sigma_2,$$

where $y = (\log_2 x)^4$.

Now we prove that $\Sigma_2 \ll x$. For any K

$$\sum_{\substack{n \leq x \\ n=Km}} d_3(n) \leq \sum_{\substack{n \leq x \\ n=Km}} d_2(n) \leq \sum_{\substack{m \leq \frac{x}{K}}} \{d(k) + d(k_1)U(m)\} \ll \frac{xd(k)}{K} \log_2 x,$$

further

$$\sum_{K > y} \frac{d(k)}{K} \ll (\log_2 x)^{-2},$$

whence $\Sigma_2 \ll x$ follows.

For the sake of brevity let $k_2 = d(k_1)$. The sum Σ_1 can be written as follows:

$$\begin{aligned}\Sigma_1 &= \sum_{K \equiv y} \sum_{\substack{(m, K)=1 \\ m \equiv \frac{x}{K}}} |\mu(m)| d(d(k) + k_2 U(m)) = \sum_{K \equiv y} \sum_{r=1}^{\infty} d(k_2(\beta + 1 + r)) P\left(\frac{x}{K}, K, r\right) = \\ &= \sum_{K \equiv y} \sum_{r \in L_x} + \sum_{K \equiv y} \sum_{r \notin L_x} = \Sigma_3 + \Sigma_4,\end{aligned}$$

where L_x denotes the interval

$$L_x = \left[\frac{1}{2} \log_2 x, \frac{3}{2} \log_2 x \right].$$

Now we prove that $\Sigma_4 \ll x$. Really

$$\begin{aligned}\sum_{\substack{U(m) \notin L_x \\ d_3(n)}} d_3(n) &\ll \sum_{K < y} d(k) \sum_{\substack{m \equiv \frac{x}{K} \\ U(m) \notin L_x}} U(m) \ll (\log_2 x)^{-1} \sum_{K \equiv y} d(k) \sum_{\substack{m \equiv \frac{x}{K} \\ U(m) \notin L_x}} (U(m) - \log_2 x)^2 \ll \\ &\ll x \sum_{K \equiv y} \frac{d(k)}{K} \ll x.\end{aligned}$$

Here we have used the inequality of TURÁN [7] stating that

$$\sum_{m \equiv x} (U(m) - \log_2 x)^2 \ll x \log_2 x.$$

Let now

$$L_x = T_x(c) + I_x(c) + R_x(c),$$

where

$$T_x(c) = [\tfrac{1}{2} \log_2 x, \log_2 x - c (\log_2 x)^{1/2}]$$

$$R_x(c) = [\log_2 x + c (\log_2 x)^{1/2}, \tfrac{3}{2} \log_2 x]$$

and correspondingly let

$$\Sigma_3 = \sum_{K \equiv y} \sum_{r \in T_x} + \sum_{K \equiv y} \sum_{r \in I_x} + \sum_{K \equiv y} \sum_{r \in R_x} = \Sigma_T + \Sigma_I + \Sigma_R.$$

Let further

$$t_l = \frac{1}{2} \log_2 x + l (\log_2 x)^{1/2}, \quad r_l = \frac{3}{2} \log_2 x - l (\log_2 x)^{1/2},$$

for $l = 1, 2, \dots, l_0$, where $l_0 = [(\log_2 x)^{1/2} - c]$, and

$$T_{x,l} = [t_{l-1}, t_l], \quad R_{x,l} = [r_{l-1}, r_l].$$

We shall prove, that the sums Σ_T and Σ_R are $o(x \log_3 x)$, whenever $c = c_x$ tends to infinity as $x \rightarrow \infty$. For this we need the Hardy—Ramanujan inequality stating that

$$P(x, K, r) \leq P(x, r) \ll \frac{x}{\log x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!}$$

for $r \in L_x$.

For any $K \leq y$ and $l \leq l_0$

$$\begin{aligned} \sum_{r \in T_{x,l}} d(k_2(\beta+1+r)) P\left(\frac{x}{K}, K, r\right) &\ll \frac{xk_2}{K \log x} \sum_{r \in T_{x,l}} d(\beta+1+r) \frac{(\log_2 x + c)^{r-1}}{(r-1)!} \ll \\ &\ll \frac{xk_2}{K \log x} \frac{(\log_2 x + c)^{t_l-1}}{(t_l-1)!} \sum_{r \in T_{x,l}} d(\beta+1+r). \end{aligned}$$

Using Lemma 6 with $A=1$ we have

$$\sum_{r \in T_{x,l}} d(\beta+1+r) = \Delta(t_l + \beta + 1) - \Delta(t_{l-1} + \beta + 1) \ll (\log_2 x)^{1/2} (\log_3 x).$$

Using the monotonicity of $\frac{(\log_2 x + c)^{t-1}}{(t-1)!}$ we have

$$\begin{aligned} \Sigma_T &\ll x (\log_2 x)^{1/2} \log_3 x \sum_{K \leq y} \frac{k_2}{K} \frac{1}{(\log_2 x)^{1/2}} \sum_{r \in T_x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!} \ll \\ &\ll x \log_3 x \sum_{r \in T_x} \frac{(\log_2 x + c)^{r-1}}{(r-1)!} = o(x \log_3 x), \end{aligned}$$

when $c_x \rightarrow \infty$ as $x \rightarrow \infty$.

It can be seen similarly, that

$$\Sigma_R = o(x \log_3 x)$$

when $c_x \rightarrow \infty$ as $x \rightarrow \infty$.

Now we start to investigate the sum Σ_I . It follows from Lemma 8, that

$$\Sigma_I = (1 + o(1)) \frac{6}{\pi^2} \frac{x}{\log x} \sum_{K \leq y} \frac{\prod_{p|K} \left(1 + \frac{1}{p}\right)^{-1}}{K} \sum_{r \in I_x(c)} d(k_2(\beta+1+r)) \frac{(\log_2 x)^{r-1}}{(r-1)!}.$$

Put

$$\Sigma_K \stackrel{\text{def}}{=} \sum_{r \in I_x(c)} d(k_2(\beta+1+r)) \frac{(\log_2 x)^{r-1}}{(r-1)!}$$

and let

$$V = (\log_2 x)^{0.4}.$$

It is easy to show that

$$\frac{(\log_2 x)^{r-1}}{(r-1)!} = (1 + o(1)) \frac{1}{V} \sum_{v=0}^{V-1} \frac{(\log_2 x)^{r+v-1}}{(r+v-1)!}$$

uniformly for every $r \in I_x(c_x)$, whenever c_x tends suitable slowly to infinity.

Using partial summation we have

$$\begin{aligned} V \Sigma_K &= (1 + o(1)) \sum_{r \in I_x(c)} \frac{(\log_2 x)^{r-1}}{(r-1)!} \sum_{v=r-V}^r d(k_2(\beta+1+v)) = \\ &= (1 + o(1)) \frac{6}{\pi^2} \sum_{r \in I_x(c)} \frac{(\log_2 x)^{r-1}}{(r-1)!} \{ \Delta_{k_2}(\beta+1+r) - \Delta_{k_2}(\beta+1+r-V) \}. \end{aligned}$$

Let us now assume, that

$$(S) \quad k_2 = d(k_1) \ll (\log_3 x)^4$$

Then using Lemma 7 with $A = k_2$ we obtain that

$$\begin{aligned} \Delta_{k_2}(\beta + 1 + r) - \Delta_{k_2}(\beta + 1 + r - V) &= c_0(k_2) V \log r + O(c_1(k_2)V) + O(V) = \\ &= c_0(k_2) V \log_3 x + O(d(k_2)(\log k_2)^2 + d(k_2)c_x(\log_2 x)^{-1/2}) \cdot V = \\ &= c_0(k_2) V \log_3 x + O(d(k_2)(\log_4 x)^2 V) = c_0(k_2)(1 + o(1)) V \log_3 x, \end{aligned}$$

whenever $c_x = O(\log_3 x)$, say.

Using the relation

$$\sum_{r \in I_x(c_x)} \frac{(\log_2 x)^{r-1}}{(r-1)!} = (1 + o(1)) \log x \quad \text{for } c_x \rightarrow \infty,$$

we have

$$\Sigma_K = c_0(k_2)(1 + o(1)) \frac{6}{\pi^2} \log x \cdot \log_3 x \quad \text{for } c_x \rightarrow \infty.$$

Further for every K by $d(mn) \leq d(m)d(n)$

$$\begin{aligned} \Delta_{k_2}(\beta + 1 + r) - \Delta_{k_2}(\beta + 1 + r - V) &\leq d(k_2)(\Delta_1(\beta + 1 + r) - \Delta_1(\beta + 1 + r - V)) \ll \\ &\ll d(k_2) V \log_3 x \end{aligned}$$

follows. Therefore for the K -s not satisfying the condition (S) we have

$$\Sigma_K \ll \frac{x k_2 d(k_2)}{(\log_3 x)^3}.$$

From these inequalities it follows rapidly that

$$\Sigma_I = (1 + o(1)) c_3 x \log_3 x,$$

where

$$c_3 = \frac{6}{\pi^2} \sum_{K \in \mathfrak{H}} \frac{c_0(k_2) \prod_{p|K} \left(1 + \frac{1}{p}\right)^{-1}}{K}.$$

Combining our results the assertion of Theorem 2 immediately follows.

References

- [1] R. BELLMAN and H. SHAPIRO, On a problem in additive number theory, *Ann. of Math.* **49** (1948), 333—340.
- [2] YU. V. LINNIK, Dispersion method for additive binär problems (in Russian).
- [3] P. ERDŐS, On the integers having exactly prime factors, *Ann. of Math.* **49** (1948), 53—66.
- [4] J. KUBILIUS, Probabilistic methods in number theory (in Russian), *Vilnius*, 1962.
- [5] H. HALBERSTAM, On the distribution of additive number-theoretic functions (III), *J. London Math. Soc.* **31** (1956), 14—27.
- [6] E. C. TITCHMARSH, The theory of the Riemann zeta-function, *Oxford*, 1951.
- [7] P. TURÁN, Über einige Verallgemeinerungen eines Satzes von Hardy und Ramanujan, *J. London Math. Soc.* **11** (1936), 125—133.

(Received May 10, 1966.)