

On the density of certain sequences of integers

By I. KÁTAI and J. MOGYORÓDI (Budapest)

1. Let $p, p_1, p_2, \dots$ be prime numbers. We use the $\ll$ symbol in Vinogradov's sense. Let $d_k(n)$ denote the number of solutions of $n = x_1 \dots x_k$, $d_2(n) = d(n)$, and put $\sigma_a(n) = \sum_{d|n} d^a$. Let ε denote arbitrarily small positive constants, not necessarily the same at every case.

For a general natural number n let B_n denote the set of those integers, all prime factors of which divide n . We call K a square-full number, if all its prime factors occur at least on the second power. In other words the number K having the prime decomposition $K = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ is square-full, if $\alpha_1 \geq 2, \dots, \alpha_r \geq 2$. Let B denote the set of all square-free numbers.

An arbitrary integer n can be written in the form

$$(1.1) \quad n = a_n \cdot b_n,$$

where $a_n \in B$, and b_n is a square-free number coprime to a_n . This representation is unique. We shall call a_n the quadratic part and b_n the square-free part of n .

Let F denote the set of those arithmetical functions, the values of which depend only on the quadratic part of the number. In other words $f(n) \in F$, if $f(n) = f(a_n)$ for all $n \geq 1$.

Here we are interested in the local-distribution on some special subsets of integers of the values of functions belonging to F .

The first result of this type is due to A. RÉNYI [1]. He proved the following assertion. If we define $f(n)$ as $f(p_1^{\alpha_1} \dots p_r^{\alpha_r}) = (\alpha_1 - 1) + \dots + (\alpha_r - 1)$ ($p_1, \dots, p_r$ are different prime numbers), then $x^{-1} N\{n \leq x, f(n) = q\}$ tends to a limit d_q for all $q = 0, 1, 2, \dots$, and $\sum d_q = 1$. This theorem was improved and generalized by many authors (see [2], [3], [4], [5]).

Suppose that $f(n) \in F$ and let $\lambda_1, \lambda_2, \dots$ be the different values taken on by $f(n)$. One of us proved the following assertion [6].

Theorem. $N\{n \leq x; f(n) = \lambda_i\} = d_i x + O(\sqrt{x} (\log x) \theta_i)$ as $x \rightarrow \infty$, where $\sum d_i = 1$, $0 \leq \theta_i \leq 1$ and $\sum \theta_i \leq 1$. The constant implied by the O -term is an absolute one.

2. Let $K_1, K_2 \in \mathcal{B}$ and let $B(x; K_1, K_2)$ be the number of $n \leq x$, for which the quadratic part of n is K_1 , and that of $n+1$ is K_2 , i.e.

$$(2.1) \quad B(x; K_1, K_2) = \sum_{n \leq x} 1 \quad (a_n = K_1, a_{n+1} = K_2).$$

Let

$$(2.2) \quad \tau(K_1, K_2) = \begin{cases} \prod_{p|K_1 K_2} \left(1 - \frac{1}{p}\right) \prod_{p \nmid K_1 K_2} \left(1 - \frac{2}{p^2}\right) & \text{when } (K_1, K_2) = 1, \\ 0 & \text{when } (K_1, K_2) > 1. \end{cases}$$

Let

$$(2.3) \quad \Delta(x) = \sum_{K_1, K_2 \in \mathcal{B}} \left| B(x; K_1, K_2) - \frac{\tau(K_1, K_2)}{K_1 K_2} x \right|.$$

First we prove the following

Theorem 1.

$$(2.4) \quad B(x; K_1, K_2) = \frac{\tau(K_1, K_2)}{K_1 K_2} x + O\left(\frac{x^{2/3}(\log x)}{(K_1 K_2)^{1/3}} d(K_1 K_2)\right).$$

Furthermore

$$(2.5) \quad \Delta(x) \ll x^{\frac{6}{7} + \varepsilon}$$

for all fixed $\varepsilon > 0$.

Let $f_1(n), f_2(n)$ be arbitrary functions belonging to F , with the set of values $\{\lambda_i\}, \{\mu_j\}$, respectively. Let

$$E(x) = \sum_{i,j} |N\{n \leq x; f_1(n) = \lambda_i, f_2(n+1) = \mu_j\} - d_{i,j} x|,$$

where

$$d_{i,j} = \sum_{K_1, K_2 \in \mathcal{B}} \frac{\tau(K_1, K_2)}{K_1 K_2} \quad (f_1(K_1) = \lambda_i, f_2(K_2) = \mu_j).$$

From Theorem 1 it follows immediately that $E(x) \leq \Delta(x) \ll x^{6/7 + \varepsilon}$.

3. Let $g(n)$ be an irreducible polynomial over the rational field with integer coefficients. Let $B(x; K) = B(x; K, g)$ denote the number of those $n \leq x$ for which the quadratic part of $g(n)$ is K . Let $\varrho(m)$ denote the number of solutions of the congruence $g(n) \equiv 0 \pmod{m}$. It is known that $\varrho(p^\alpha) \ll 1$ for $\alpha = 1, 2, \dots$ uniformly in p and that $\varrho(m)$ is a multiplicative function. Let

$$(3.1) \quad \tau(K) = \prod_{p|K} \left(1 - \frac{\varrho(p)}{p}\right) \prod_{p \nmid K} \left(1 - \frac{\varrho(p)}{p^2}\right)$$

and

$$(3.2) \quad P(x) = \sum_{K \in \mathcal{B}} \left| B(x; K) - \frac{\tau(K)}{K} x \right|.$$

It seems likely that

$$(3.3) \quad x^{-1} P(x) \rightarrow 0 \quad \text{as } x \rightarrow \infty,$$

for all polynomials. For the moment we can prove this only for polynomials of degree not higher than 3.

Theorem 2. Let $g(n) = n^2 + 1$. Then we have

$$(3.4) \quad B(x; K) = \frac{\tau(K)}{K} x + O\left(\frac{x^{2/3} \log x}{K^{1/3}} d(K)\right)$$

and

$$(3.5) \quad P(x) \ll x^{\frac{6}{7} + \varepsilon},$$

where ε is an arbitrary positive constant.

We state without proof the

Theorem 3. Let $g(n)$ be an irreducible polynomial of degree 3. Then

$$P(x) = o(x).$$

4. Proof of Theorem 1. First we prove (2.4). Since for $(K_1, K_2) > 1$ we have $B(x; K_1, K_2) = 0$, thus (2.4) holds in this case. Assume now that $(K_1, K_2) = 1$. Let d_i run over the sets of integers relatively prime to K_i , and δ_i the set $\mathcal{B}_{K_i}$ (resp. for $i = 1, 2$). Using the relations

$$\sum_{\delta_1 d_1^2 K_1 | n} \mu(\delta_1) \mu(d_1) = \begin{cases} 1, & \text{if } a_n = K_1, \\ 0 & \text{otherwise;} \end{cases}$$

$$\sum_{\delta_2 d_2^2 K_2 | n+1} \mu(\delta_2) \mu(d_2) = \begin{cases} 1, & \text{if } a_{n+1} = K_2, \\ 0 & \text{otherwise,} \end{cases}$$

we have

$$B(x; K_1, K_2) = \sum_{n \leq x} \sum \mu(\delta_1 \delta_2 d_1 d_2),$$

where the second Σ means a summation over those $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1 d_1^2 K_1 | n, \delta_2 d_2^2 K_2 | n+1$. By changing the order of the summation we obtain

$$(4.1) \quad B(x; K_1, K_2) = \sum_{\substack{\delta_1, \delta_2 \\ d_1, d_2}} \mu(\delta_1 \delta_2 d_1 d_2) S(x; K_1 \delta_1 d_1^2, K_2 \delta_2 d_2^2),$$

where $S(x; a, b)$ is the number of those $n \leq x$ for which $n \equiv 0 \pmod{a}$ and $n+1 \equiv 0 \pmod{b}$ hold. This congruence system is solvable only if $(a, b) = 1$, and for $(a, b) = 1$

$$(4.2) \quad S(x; a, b) = \frac{x}{ab} + O(1).$$

Thus we deduce from (4.1)

$$B(x; K_1, K_2) = \frac{x}{K_1 K_2} \Sigma' + O(\Sigma_{K_1, K_2}^{(1)}) + O(\Sigma_{K_1, K_2}^{(2)}),$$

where

$$(4.3) \quad \Sigma' = \sum \frac{\mu(\delta_1 \delta_2 d_1 d_2)}{\delta_1 \delta_2 d_1^2 d_2^2}.$$

The summation in (4.3) is extended over those $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1 \delta_2 d_1^2 d_2^2 \leq x^\beta$ (where β is a constant satisfying the relation $1 \leq \beta < 2$). $\Sigma_{K_1, K_2}^{(1)}$ is the number of the values $\delta_1, \delta_2, d_1, d_2$ satisfying $\delta_1 \delta_2 d_1^2 d_2^2 K_1 K_2 \leq x^\beta$, and $\Sigma_{K_1, K_2}^{(1)}$ denotes the number of $\delta_1, \delta_2, d_1, d_2$ for which $\delta_1 \delta_2 d_1^2 d_2^2 K_1 K_2 > x^\beta$ and $\delta_1 d_1^2 K_1 |n, \delta_2 d_2^2 K_2 |n+1$ for one $n \leq x$ at least.

Taking into account that

$$\sum \frac{\mu(\delta_1 \delta_2 d_1 d_2)}{\delta_1 \delta_2 d_1^2 d_2^2} = \tau(K_1, K_2),$$

we deduce from (4.3)

$$\Sigma' = \tau(K_1, K_2) + O(\Sigma_{K_1, K_2}^{(3)}),$$

where

$$(4.4) \quad \Sigma_{K_1, K_2}^{(3)} = \sum \frac{1}{\delta_1 \delta_2 d_1^2 d_2^2} \quad (\delta_1 \delta_2 d_1^2 d_2^2 K_1 K_2 > x^\beta).$$

Using that $\sum_{v \leq u} d(v) v^{-2} \ll u^{-1} \log u$ and that $v = d_1 d_2$ has at most $d(v)$ solutions in d_1, d_2 , we deduce from (4.4) that

$$(4.5) \quad \Sigma_{K_1, K_2}^{(3)} \ll x^{-\beta/2} (\log x) \sqrt{K_1 K_2} \sigma_{-1/2}(K_1 K_2).$$

In order to estimate $\Sigma_{K_1, K_2}^{(3)}$, we consider that the number of d_1, d_2 satisfying $\delta_1 \delta_2 d_1^2 d_2^2 K_1 K_2 \leq x^\beta$ is smaller than

$$\sum_{v \leq N} d(v) \ll N \log x, \quad N = \frac{x^{\beta/2}}{\sqrt{\delta_1 \delta_2 K_1 K_2}},$$

whence after a summation over the δ 's we obtain

$$(4.6) \quad \Sigma_{K_1, K_2}^{(1)} \ll \frac{x^{\beta/2} \log x}{\sqrt{K_1 K_2}} \sigma_{-1/2}(K_1 K_2).$$

For the estimation of $\Sigma_{K_1, K_2}^{(2)}$ we need the following

Lemma 1. *Let a, b arbitrary positive integers. Then the number of solutions $1 \leq u, v \leq x$ of the equation*

$$au^2 - bv^2 = 1$$

is at most $O(\log x)$.

Let

$$(4.7) \quad n = l_1 \delta_1 K_1 d_1^2, n+1 = l_2 \delta_2 K_2 d_2^2$$

and let $R(\delta_1, \delta_2)$ denote the number of those $n \leq x$ for which (4.7) is satisfied with suitable l_1, l_2, d_1, d_2 satisfying the inequality $\delta_1 \delta_2 K_1 K_2 d_1^2 d_2^2 > x^\beta$. For fixed $l_1, l_2, \delta_1, \delta_2$ the number of the n 's in (4.7) is at most $O(\log x)$ by Lemma 1. Hence by $l_1 l_2 (\leq 2x(\delta_1 \delta_2 K_1 K_2 d_1^2 d_2^2)^{-1}) \leq 2x^{2-\beta}$ we have

$$R(\delta_1, \delta_2) \ll (\log x) \sum_{l_1 l_2 \leq 2x^{2-\beta}} 1 \ll x^{2-\beta} \log^2 x$$

Summing over the δ 's we obtain

$$(4.8) \quad \Sigma_{K_1, K_2}^{(2)} \ll x^{2-\beta} \log^2 x d(K_1 K_2).$$

Now we choose β as follows. If $K_1 K_2 = x^\gamma$, then $\beta = \frac{\gamma+4}{3}$. With this β we have

$$\frac{x^{\beta/2}}{\sqrt[3]{K_1 K_2}} = x^{2-\beta} = \frac{x^{2/3}}{(K_1 K_2)^{1/3}}.$$

Thus we obtain (2. 4) immediately by combining our inequalities (4. 8), (4. 6), (4. 4). In order to prove (2. 5) we remark that $\Delta(x) \equiv O(\Sigma^{(1)}) + O(\Sigma^{(2)}) + O(\Sigma^{(3)})$, where

$$\Sigma^{(1)} = \sum_{K_1, K_2 \in \mathcal{B}} \Sigma_{K_1, K_2}^{(1)}, \quad \Sigma^{(2)} = \sum_{K_1, K_2 \in \mathcal{B}} \Sigma_{K_1, K_2}^{(2)},$$

$$\Sigma^{(3)} = \sum_{K_1, K_2 \in \mathcal{J}} \frac{x}{K_1 K_2} \Sigma_{K_1, K_2}^{(3)}.$$

Using $\sum_{K \leq x} \frac{\sigma_{-1/2}(K)}{\sqrt{K}} \ll \log x$ we have from (4. 6) and (4. 4)

$$(4. 9) \quad \Sigma^{(1)} \ll x^{1-\beta/2} (\log x)^3, \quad \Sigma^{(3)} \ll x^{\beta/2} (\log x)^3.$$

Now we consider $\Sigma^{(2)}$. Set $N_1 = \delta_1 d_1^2 K_1$, $N_2 = \delta_2 d_2^2 K_2$. In our case $N_1 N_2 > x^\beta$ and N_i has at most $d_3(N_i)$ representations as product of δ_i , d_i^2 , K_i . Let $N_i = u_i^2 v_i$, where u_i^2 is the greatest quadratic divisor of N_i . Using the fact that N_i is a square-full number, we have $v_i \leq N_i^{1/3}$. Taking $n = l_1 N_1 = l_1 v_1 u_1^2$, $n+1 = l_2 v_2 u_2^2$ we have that $\Sigma^{(2)} \ll x^\varepsilon R$, where R denotes the number of solutions of the equation

$$(4. 10) \quad l_2 v_2 u_2^2 - l_1 v_1 u_1^2 = 1$$

for those $l_1, l_2, v_1, v_2, u_1, u_2$ which satisfy the inequality

$$l_1 v_1 l_2 v_2 \left(\leq 2 \frac{x^2}{u_1^2 u_2^2} \leq 2 \frac{x^2}{(N_1 N_2)^{2/3}} \right) \leq 2x^{2-\frac{2}{3}\beta}, \quad u_1 \leq x, \quad u_2 \leq x.$$

By Lemma 1 (4. 10) has at most $O(\log x)$ solutions in u_1, u_2 for fixed l_1, l_2, v_1, v_2 . Thus

$$R \ll (\log x) \sum_{l_1 l_2 v_1 v_2 \leq 2x^{2-\frac{2}{3}\beta}} 1 \ll x^{2-\frac{2}{3}\beta+\varepsilon},$$

and consequently

$$(4. 11) \quad \Sigma^{(2)} \ll x^{2-\frac{2}{3}\beta+\varepsilon}.$$

Now we choose $\beta = \frac{12}{7}$. Taking into account (4. 11), (4. 9) we deduce (2. 5.)

5. PROOF OF THEOREM 2. This is a similar one to that of Theorem 1, therefore we give only a scetch for it. It is known that $\varrho(p^\alpha) = 2$ or 0 , according to $p \equiv 1$ or $\equiv -1 \pmod{4}$. Furthermore $\varrho(2) = 1$ and $\varrho(2^\alpha) = 0$ for $\alpha \geq 2$.

Let δ run over the divisors of K , and let d denote the numbers coprime to K . Then

$$(5. 1) \quad B(x; K) = \sum_{\delta, d} \mu(\delta d) \varrho_x(\delta d^2 K),$$

where $\varrho_x(m)$ denotes in general those number of $n \leq x$ for which m divides $n^2 + 1$. Then we have

$$(5.2) \quad \varrho_x(m) = \frac{\varrho(m)x}{m} + O(\varrho(m)).$$

From (5.1) and (5.2) we have

$$(5.3) \quad B(x; K) = \frac{\tau(K)}{K} x + O\left(\frac{x}{K} \Sigma_K^{(3)}\right) + O(\Sigma_K^{(1)}) + O(\Sigma_K^{(2)}),$$

where

$$(5.4) \quad \Sigma_K^{(1)} = \sum_{\delta, d} \varrho(\delta K d^2) \quad (\delta d^2 K \leq x^\beta)$$

$$(5.5) \quad \Sigma_K^{(2)} = \sum_{\delta, d} \varrho_x(\delta K d^2) \quad (\delta d^2 K > x^\beta)$$

$$(5.6) \quad \Sigma_K^{(3)} = \sum_{\delta, d} \frac{\varrho(\delta K d^2)}{\delta K d^2} \quad (\delta d^2 K \leq x^\beta).$$

Using that $\varrho(\delta K) = \varrho(K)$, $\varrho(d^2) = \varrho(d)$ and that $\sum_{m \leq y} \varrho(m) \ll y$ we have without any difficulty, that

$$(5.7) \quad \Sigma_K^{(1)} \ll \frac{x^{\beta/2}}{\sqrt{K}} \varrho(K) \sigma_{-1/2}(K),$$

$$(5.8) \quad \Sigma_K^{(3)} \ll x^{-\beta/2} \sqrt{K} \sigma_{-1/2}(K).$$

For the estimation of $\Sigma_K^{(2)}$ we use Lemma 1. $\Sigma_K^{(2)}$ is not greater than the number of the solutions of

$$(5.9) \quad n^2 - l \delta K d^2 = 1 \quad (n \leq x)$$

for those d, n, l, δ which satisfy the inequality $l \ll x^{2-\beta}$. For fixed l, δ (5.9) has at most $O(\log x)$ solutions. Consequently

$$(5.10) \quad \Sigma_K^{(2)} \ll x^{2-\beta} (\log x) d(K).$$

Choosing $\beta = \frac{\gamma+4}{3}$, where $K = x^\gamma$, and taking into account (5.3), (5.7), (5.8), (5.10) we obtain (3.4).

The estimation of $P(x)$ goes in a similar manner as that of $A(x)$ and thus we drop it.

6. The proof of Theorem 3 goes in a similar way as that of Theorem 2 applying the following lemma due to C. HOOLEY [8].

Lemma 2. *If $g(n)$ is an irreducible polynomial of degree 3, then the number of $n \leq x$, for which there exists a p^2 divisor of $g(n)$, $> \log x$ is at most*

$$O(x (\log x)^{-A/\log \log \log x}),$$

$A > 0$ is a constant.

References

- [1] A. RÉNYI, On the density of certain sequences of integers, *Publ. Inst. Math. Belgrád*, **8** (1955), 157—162.
- [2] H. DELANGE, Sur un theoreme de Rényi, *Acta Arithm.*, **11** (1965), 241—252.
- [3] I. KÁTAI, Egy megjegyzés H. Delange „Sur un theoreme de Rényi” c. dolgozatához, *MTA III; Oszt. Közl.*, **16** (1966), 269—273.
- [4] H. DELANGE, Sur un theoreme de Rényi II, *Acta Arithm.*, **13** (1968), 339—362.
- [5] И. П. Кубильюс, Вероятностные методы в теории чисел, Вильнюс, 1962.
- [6] I. KÁTAI, Egy lokális határeloszlástétel a számelméletben, *Mat Lapok* (in print).
- [7] F. V. ATKINSON and LORD CHERWELL, On arithmetical functions, *Quarterly J. Math. (Oxford)*, **20** (1949), 65—79.
- [8] C. HOOLEY, On the power free values of polynomials, *Mathematika*, **14** (1967), 21—26.

(Received September 12, 1966.)