

On Archimedean ring extensions

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1. Introduction. In literature there are results showing the preservation of Archimedean character of ordering in some ring extensions. For example, the quotient fields of Archimedean rings, the finite extensions of Archimedean fields and semi-simple algebras over Archimedean fields are again Archimedean [1; 127].

In this note we consider the algebraic algebras over Archimedean subrings which are not necessarily fields, and discuss whether they can be Archimedean. This problem is tackled via weak Archimedean rings. It is proved that if a fully ordered (f.o.) ring with identity is algebraic over an Archimedean subfield, then it is only weak Archimedean but not necessarily Archimedean. But if the above ring is an integral domain, then it is certainly an Archimedean ring. The Archimedean nature of f.o. division ring is shown to be completely determined by the algebraic nature of its bounded elements. Furthermore, we shall prove that a commutative integral domain which is an algebraic algebra over Archimedean Noetherian subring is an Archimedean ring.

1.1 Definition. A fully ordered (f.o.) ring R with identity is said to be weak Archimedean if every element in R is bounded. An element x in R is said to be bounded (by the ring of integers I) if $|x|$ is less than some positive integral multiple of the identity.

1.2 Definition. A f.o. ring R with identity is said to be an algebraic algebra over a subring F of R or algebraic over F if every element of R is algebraic over F . An element $x \in R$ is said to be algebraic over F if either $\sum_{i=0}^n a_i x^i = 0$ or $\sum_{i=0}^n x^i a_i = 0$ for some $a_i \in F$. If $a_n = 1$, we say that x is integral over F .

1.3 Notation. Throughout this paper all rings are associative rings with identity and Archimedean f.o. rings are defined as in [1].

2. Extensions of archimedean rings

2.1 Theorem. Let R be a f.o. ring with identity such that R is algebraic over a division subring F . Then R is a local ring with the unique maximal right ideal as a nil ideal.

PROOF. Let R be an integral domain. If $x \neq 0$ and $x \in R$, then $\sum_{r=0}^n a_r x^r = 0$, where $a_i \in F$ and n is the minimal degree of the polynomial satisfied by x .

Let $y = a_n x^{n-1} + \dots + a_1 \neq 0$. Since R is an integral domain, $yx = -a_0 \neq 0$. Hence $(-a_0)^{-1}yx = 1$. Also $x(-a_0)^{-1}y = 1$, since, otherwise $[x(-a_0)^{-1}y - 1]x = 0$, a contradiction. Similarly x has a two-sided inverse if $\sum_{r=0}^n x^r a_r = 0$. Thus R is a division ring, a trivial local ring. If R is not an integral domain, then the set N of all nilpotent elements in R is a non-zero convex ideal [1, 130]. So R/N is a f.o. integral domain with identity algebraic over $\bar{F}$, the image of F in R/N . Hence from the above R/N is a division ring. Thus R is a local ring with N as the unique maximal right ideal of R .

2.2 Lemma. *Let R be a f.o. ring with identity and suppose $x \geq 0$. Then if $\sum_{r=0}^n a_r x^r = 0$ with the coefficients in a subring F and if $a_n \geq 1$, x is bounded by an element in F .*

PROOF. Let $a_n = 1 + t$, $t \geq 0$. There exists an $c > 0$ such that $c \in F$ and $c \geq 1 - a_j$, $j = 0, 1, 2, \dots, n$. The proof is completed by showing $x < c$. Suppose on the contrary $x \geq c$. Then $a_j \geq 1 - c \geq 1 - x$ and hence $a_j x^j \geq (1 - x)x^j$ for $j = 1, 2, \dots, n$. Therefore $0 = a_n x^n + \dots + a_0 \geq (1 + t)x^n + \dots + (1 - x) = 1 + tx^n > 1$, a contradiction.

2.3 Theorem. *Let R be a f.o. ring with identity. Then R is weak Archimedean if either (i) R is integral over a weak Archimedean subring or (ii) R is algebraic over an Archimedean subring F .*

PROOF. The first case is an immediate consequence of the lemma 2.2. In the second case, let $x \in R \setminus F$. Then $a_n x^n + \dots + a_0 = 0$, $a_i \in F$. We can always assume that $a_n \geq 1$. Otherwise, if $a_n < 1$, there exists a positive integer m such that $ma_n > 1$. So $ma_n x^n + \dots + ma_0 = 0$. Now the result follows from 2.2.

2.4 Lemma. *If a division ring D is a weak Archimedean f.o. ring, then D is Archimedean.*

PROOF. Let a and b be any two non-zero elements of D . Suppose $a, b > 0$. By hypothesis, $b < m$ and $a^{-1} < n$ for some positive integers m and n . Then follows $ba^{-1} < N$, N being a positive integer. So $b < Na$ and thus D is Archimedean.

Combining the theorems 2.1 and 2.3 we have

2.5 Corollary. *Let R be a f.o. ring with identity and suppose R is algebraic over a weak Archimedean division subring (equivalently Archimedean subfield). Then R is a weak Archimedean local ring with the unique maximal right ideal as a nil ideal.*

It is stated in the proposition 3 of [1; 127] that every algebraic algebra over an Archimedean subfield is Archimedean. This need not be true generally unless the algebraic algebra is an integral domain, as can be seen in the following example.

2.6 Example. Let $S = \{a + bx, a, b \in R, \text{ the rational number field}\}$. Suppose $x^2 = 0$ and $dx = xd$ for every $d \in R$. Then S is a ring with identity under the usual operations as in polynomials. S is a f.o. ring by setting $a + bx > 0$ if $a > 0$ or $b > 0$ if $a = 0$. Since $(a + bx)^2 - 2a(a + bx) + a^2 = 0$, S is algebraic over R . But S is not Archimedean since it has zero-divisors.

But by combining 2. 4 and 2. 5 we obtain,

2. 7 Corollary. Let R be a f.o. integral domain with identity. Then R is an Archimedean ordered field if R is algebraic over an Archimedean subfield.

It is possible to determine the Archimedean character of a division ring by knowing the algebraic nature of bounded elements. So we begin by proving an interesting result.

2. 8 Definition. A ring D with identity is called a valuation ring with respect to a skewfield Q , if Q is a two-sided quotient skewfield of D and if $x \in Q$ then x or $x^{-1} \in D$. This is equivalent to saying that the right ideals and also the left ideals are linearly ordered by set inclusion.

2. 9 Definition. A ring Q with identity is called a right quotient ring of a ring R with identity if i) $R \subseteq Q$, ii) every non-zero divisor in R is invertible in Q and iii) every $q (\neq 0) \in Q$ is of the form ab^{-1} , $a, b \in R$ and b is a non-zero divisor in R .

The Archimedean character is always preserved under passage to the (right) quotient fields where as the weak Archimedean character is not all carried to the right quotient skewfields in the non-commutative case, since otherwise, the quotient skewfield becomes Archimedean by lemma 2. 4 and hence the original ring becomes commutative.

2. 10 Proposition. Let D be the set of all bounded elements in a f.o. division ring Q . Then D is a valuation ring with Q as a two-sided quotient skewfield and D is the maximal weak Archimedean subring of Q .

PROOF. Evidently D is a subring with identity same as that of Q . Let $x > 0$ be in Q and $x \notin D$. Then $x > 1$, which implies $x^{-1} \leq 1$. Thus $x^{-1} \in D$. If $x \in Q \setminus D$, then $x = x^{-1}[(x^2)^{-1}]^{-1} = [(x^2)^{-1}]^{-1}x^{-1}$, where $x^{-1} \in D$. Also $(x^2)^{-1} \in D$, since, otherwise, $(x^2)^{-1}$ is unbounded. Then x^2 is bounded. So $x = x^2x^{-1} \in D$. Thus D is a valuation ring with Q as a two-sided quotient skewfield.

2. 11 Theorem. Let D be the set of all bounded elements in an ordered division ring Q . Then Q is Archimedean if D is algebraic over an Archimedean subring S with identity.

PROOF. If $D = Q$, then by 2. 4 Q is Archimedean. So assume $D \neq Q$. Since S is Archimedean, S is a commutative integral domain with identity and so has a quotient field S^* in Q . Furthermore the quotient field S^* itself can be verified to be Archimedean. Hence $S^* \subseteq D$. Now D is an integral domain with identity which has S^* as a subring and which is algebraic over an Archimedean field S^* , since D is algebraic over S and $S \subseteq S^*$. Hence by 2. 7, D is a field. But Q is the quotient field of D by 2. 10. Hence $D = Q$. Thus Q becomes Archimedean.

Now we discuss the Archimedean extensions of Archimedean Noetherian rings which need not be necessarily fields.

2. 12 Lemma. Let R be a commutative integral domain with identity and suppose that S is a Noetherian subring of R with identity. Then i) if a and b are in R and algebraic over S , ab is algebraic over S ii) if R is a finitely generated S -module, R is algebraic over S .

PROOF. Since a and b are algebraic over S , the subring $S[a, b]$ generated by S , a and b is a finitely generated S -module. But S is Noetherian and $S[ab]$, a subring generated by S and ab , is a S -submodule of $S[a, b]$. Hence $S[ab]$ is itself finitely generated S -module. Then we can write $S[ab] = Sf_1(ab) + Sf_2(ab) + \dots + Sf_t(ab)$. Choose N greater than the degrees of $\{f_i(x)\}$. Then $(ab)^N = S_1f_1(ab) + \dots + S_tf_t(ab)$ where $S_i \in S$. Thus ab is algebraic over S .

To prove (ii), by a similar argument as above, if $a \in R$, there exists N such that $a^N = S_1f_1(a) + \dots + S_tf_t(a)$ where $S_i \in S$, since $S[a]$ is a finitely generated S -module. Hence R is algebraic over S .

2.13 Theorem. *Let R be a commutative f.o. integral domain with identity. Then R is Archimedean if either i) R is algebraic over a Noetherian Archimedean subring S with identity or ii) R is a finitely generated module over a Noetherian Archimedean subring S with identity.*

PROOF. By lemma 2.12 (ii) $\Rightarrow$ (i). Now R has a quotient field Q and S has also a quotient field S^* and $S^* \subseteq Q$. Since S is Archimedean, S^* is also Archimedean. Let $x (\neq 0) \in Q$. Then $x = ab^{-1}$, $a, b \in R$. Since b is algebraic over S , b^{-1} is also algebraic over S . Then ab^{-1} is algebraic over S by lemma 2.12. Consequently Q is algebraic over the Archimedean field S^* . Hence by 2.7, Q is Archimedean. Thus R is Archimedean.

References

- [1] L. FUCHS, Partially Ordered Algebraic Systems *London*, 1963.

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