

The topological group of the p -adic integers

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Introduction. In this paper *) we characterize the topological group E_p of p -adic integers (for some prime number p) (see Theorem 1). It is also shown that E_p does not allow of any locally compact group topology other than the natural compact topology and the discrete topology (Theorem 2.). Finally all compact abelian groups with this property are characterized (Theorem 3).

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§ 1. In this section we show that four properties characterize the E_p 's. Our proof depends strongly on the duality theory of locally compact groups.

Theorem 1. *Let G be an infinite compact topological group. Then the following conditions are equivalent:*

- (1) G is abelian and every infinite collection of closed subgroups has a trivial intersection.
- (2) G is abelian and every nontrivial closed subgroup of G is open.
- (3) G is topologically isomorphic to the group E_p for some prime p .
- (4) The closed subgroups of G form a chain ordered by set-inclusion.
- (5) G is abelian and every closed subgroup is nG for some integer n .

PROOF. We will give a cyclic proof.

(1) $\Rightarrow$ (2): Let S be any nontrivial closed subgroup of G . If G/S had an infinity of nontrivial closed subgroups correspondingly G will have infinity of closed subgroups whose intersection contains S contradicting (1). Hence G/S has only a finite number of closed subgroups and so its dual is a discrete abelian group with only a finite number of subgroups since each subgroup in the dual is the annihilator of a closed subgroup of G/S . But it is easy to see that an abelian group with just a finite number of subgroups is finite and so self-dual. Thus G/S is finite and so S is open.

(2) $\Rightarrow$ (3). Let H be the character group of G and S' a proper subgroup of H . If S is the annihilator in G of S' , then by hypothesis S is open and so G/S is finite. But then S' is the dual of G/S and so S' is finite. Thus every proper subgroup of H is finite and so $H \simeq c(p^\infty)$ for some prime [1, p. 67]. Hence G (dual of H) is topologically isomorphic to E_p (the dual of $c(p^\infty)$).

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(3) $\Rightarrow$ (4). Since G is topologically isomorphic to E_p , the character group H of G is $\simeq c(p^\infty)$. But in $c(p^\infty)$ the subgroups form a chain. Hence their annihilators in G namely the closed subgroups of G form a chain, since whenever $S' \supset T'$, $S \subset T$, S, T being the annihilators of S', T' respectively [7].

(4) $\Rightarrow$ (5). If a and b are elements of G , the closures of the abelian groups $\{a\}$ and $\{b\}$ are abelian subgroups of G [5] and are comparable so that a and b commute. Hence G is abelian. If H , the character group of G , contains an element a of infinite order the subgroups $\{2a\}$ and $\{3a\}$ are not comparable and so their annihilators cannot be comparable. Thus H has to be a torsion group. The annihilator $H(n)$ of the group nG consists of all elements x' in H s.t. $nx' = 0$ [4]. Since H is torsion each of its elements belongs to some $H(n)$. The closed subgroup $\bigcap_n nG$ will be annihilated by each element of H and so $\bigcap_n nG = 0$.

Let now S be any non zero closed subgroup of G . Then S contains some nG since $S \not\subset \bigcap_n nG = 0$, and the closed subgroups form a chain. Consider the group $\bar{G} = G/nG$. This is a torsion group of bounded order any two finite subgroups are comparable. Hence if $\bar{x}$ is an element of maximum order in $\bar{G}$ then $\bar{G} = \langle \bar{x} \rangle$. Since $\bar{G}$ is cyclic, any of its subgroups is $m\bar{G}$ for some m . If x is a preimage of $\bar{x}$ then $mx \rightarrow m\bar{x}$ and so $mx \notin nG$. Hence $mG \supset nG$ and $mG \rightarrow m\bar{G}$. Hence if the closed subgroup S maps upon $m\bar{G}$ we should have $S = mG$ since there is a 1—1 correspondence between closed subgroups of G containing nG and closed subgroups of $\bar{G}$.

(5) $\Rightarrow$ (1). Let $\{S_\alpha\}$ be any collection of closed subgroups of G and let $\bigcap S_\alpha \neq 0$. Then $\bigcap S_\alpha = nG$ for some n . Since each closed subgroup of G is of the form mG , the closed subgroups of $\bar{G} = G/nG$ are $\bar{G}, 2\bar{G}, \dots, (n-1)\bar{G}$ only. Hence the closed subgroups of G containing nG are $G, 2G, \dots, (n-1)G$ only. Thus the collection $\{S_\alpha\}$ is a finite collection and so our result follows. This establishes Theorem 1.

Corollary 1. In E_p , any proper closed subgroup is of the form $p^k E_p$.

PROOF. By Theorem 1, any proper closed subgroup is of the form nG , $G = E_p$. Now G/nG is the dual of a subgroup of $c(p^\infty)$ and hence is cyclic of order p^k say. Now every element x of G/nG satisfies $nx = 0$ so that $n = p^k \cdot m$. Now $p^k G \subseteq nG$ since G/nG is of order p^k . $nG = m \cdot p^k G \subseteq p^k G$.

Hence $nG = p^k G$.

Corollary 2. In E_p , the $p^k E_p$'s, $k = 0, 1, 2, \dots$ form a basis of neighbourhoods at the identity.

PROOF. E_p , being the dual of the (infinite) discrete torsion group $c(p^\infty)$, is a (nondiscrete) compact totally disconnected group and hence [5] has a basis at the identity consisting of compact open subgroups. Then corollary 1 completes the proof.

§ 2. A. HULANICKI [2] has characterized all abelian groups which allow of a unique compact group topology, as a consequence of which it follows that for any E_p the natural topology is the only compact group topology on it. Another well known fact which we need in this section is that no E_p allows of a homomorphism into E_q for any $q \neq p$. This could be easily proved using the facts that E_p is a reduced group ($\bigcap_n nE_p = 0$), and is divisible by all integers prime to p ; but not by p [as a matter of fact no non-zero subgroup of E_p is divisible by p].

Theorem 2. The only locally compact group topologies on the group E_p of the p -adic integers are the natural compact topology and the discrete topology.

PROOF. Let (E_p, τ) be a locally compact topological group. If τ is discrete there is nothing to prove. Let τ be nondiscrete. Then by [7] $E_p = R^n + A$, where A contains a compact open subgroup S . Since E_p is a reduced group, $n=0$. Also S cannot have a connected component since a compact connected abelian group is divisible and E_p is reduced. Hence S is totally disconnected. Also S is torsion free since E_p is torsion free. Hence S is algebraically isomorphic to a complete product of some E_q 's. If one of these q 's is $\neq p$ we will have a isomorphism of some E_q into $S \subset E_p$. Since this is not possible, we have each $q=p$; so S contains a E_p .

Thus we have an endomorphism of E_p . But E_p is a compact topological integral domain and any endomorphism of the group E_p is represented by $x \rightarrow rx$, $r \in E_p$ [1, p. 212]. Under the natural compact topology this is a continuous map and hence the image is a closed subgroup of E_p and so is open by Th. 1 and hence is of finite index. Since S contains this image, S is also of finite index in E_p . Now S is τ -compact and E_p is a finite union of τ -compact sets $S, x_1S, \dots, x_nS$ and so E_p is τ -compact, i.e. τ is compact. But the only compact group topology on E_p is the natural topology and hence our result follows.

Theorem 3. Let (G, T) be a compact abelian topological group. Then the following are equivalent:

- (1) G has no locally compact group topology other than T and the discrete topology
- (2) $G = A + B$, a topological direct sum, where B is discrete finite and $A=0$ or a compact p -adic integer group E_p .

PROOF.

(2) $\Rightarrow$ (1). Let $G = E_p + F$. Suppose τ is a locally compact group topology on G . Then τ is either the 'natural' product topology or is discrete. Now $G = R^n + A$, with A having a compact open subgroup S . If F is finite of order m , then $mR^n = R^n \subset E_p$ and is a divisible subgroup of E_p . Since E_p is reduced we have $n=0$. Thus $G=A$ and S is a compact open subgroup of G . Since a compact connected group is divisible, by a similar argument, S has no connected component and hence is totally disconnected. Now mS is a compact group $\subset E_p$. If $mS=0$, it means that $S \subset F$ and hence is finite so that τ is discrete since it has a finite open subgroup. If $mS \neq 0$, by declaring this to be a open subgroup we get a locally compact group topology on E_p . This topology is not discrete because mS is a compact and infinite group and a compact discrete space is finite. So this is the natural topology on E_p . Hence E_p/mS is finite since mS is open in E_p . Since mS is τ -compact, so is E_p . But G/E_p is finite and so E_p is τ -open and on E_p the induced topology is the natural topology of E_p . But any group topology having a open subgroup is determined by the topology on the subgroup [3]. Hence τ is the product topology of E_p and F . The proof is obvious when $A=0$ since then G is finite.

(1) $\Rightarrow$ (2). Let G have T and the discrete topology as the only locally compact group topologies. Since G is compact, this is the only compact group topology on it. Hence by HULANICKI [2]. $G = \prod_p D_p$, where each D_p = product of a finite p -group and a finite number of p -adic integer groups with the product topology. If $E_p, E_q, q \neq p$ occur then by giving compact group topology to E_p and discrete to the rest

we get a new nondiscrete locally compact group topology. Hence there is at most one p for which D_p contains a E_p . Also in this D_p , E_p cannot occur more than once since by picking out one of them and declaring it to be open with its natural compact topology we get a new nondiscrete locally compact group topology. Now each D_q , $q \neq p$ is finite and if there are an infinity of q 's for which $D_q \neq 0$, then by picking a proper infinite subset J of these q 's and declaring $\prod_{q \in J} D_q$ to be open with its product topology we get again a new non discrete locally compact group topology. Hence we have $\prod D_q$, $q \neq p$ is a finite group and D_p is either finite or finite group $+$ a E_p so that the assertion follows.

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