Tensor products
and maximal subgroups of commutative semigroups

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R. O FULP defined the tensor product of commutative semigroups and began the investigation of these products in [2]. The study of these products was begun independently in [3] and continued in [4], [5], [6] and [7]. The present note *) arose from meditations on a particularly pretty result in [2] which Professor Fulp was kind enough to communicate to us in advance of publication. Fulp proved that if H and K are maximal subgroups of commutative semigroups A and B then $H \otimes K$ may be regarded as a subgroup of $A \otimes B$ in the natural way. He left unanswered the question (which he suggested to us) of whether $H \otimes K$ is a *maximal* subgroup of $A \otimes B$.

Our purpose here is to give a positive answer to this question. The answer drops out quickly from a new method of proof that $H \otimes K$ is imbedded in $A \otimes B$.

All semigroups considered are assumed to be commutative and their operations are denoted additively. *All capital letters denote commutative semigroups.* By the tensor product $A \otimes B$ of A and B we mean the quotient semigroup $F(A \times B)/\tau$ where $F(A \times B)$ is the free commutative semigroup on the set $A \times B$ and τ is the finest congruence relation for which:

$$(a_1 + a_2, b)\tau(a_1, b) + (a_2, b) \quad \text{and} \quad (a, b_1 + b_2)\tau(a, b_1) + (a, b_2)$$

hold for all $a_1, a_2, a \in A$ and $b, b_1, b_2 \in B$. It has been shown ([2], [3]) that when A and B are groups this tensor product is identifiable with the usual tensor product of A and B regarded as Abelian groups. For any further concepts that may need clarification see [1], [2], or [3].

§ 1. Let X and Y be subsemigroups of A and B. Let $i: X \to A$ and $j: Y \to B$ be the inclusion maps. We say that $X \otimes Y$ is imbedded in $A \otimes B$ if $i \otimes j: X \otimes Y \to A \otimes B$ is 1—1. When $i \otimes j$ is 1—1 we identify $X \otimes Y$ with its image and write $X \otimes Y \subset A \otimes B$.

Theorem. *If H and K are maximal subgroups of commutative semigroups A and B then $H \otimes K$ is imbedded in $A \otimes B$ and is a maximal subgroup of $A \otimes B$.*

*) This work was done in part while the author was supperted by the National Science Foundation through its Research Participation for College Teachers program at the University of Oklahoma, summer of 1967.

PROOF. We base the proof on the two imbedding propositions given below in § 2. The trick is to use Archimedean components and to play properties of subgroups against properties of ideals.

Let E and F be the Archimedean components of A and B which contain H and K. Let $U=\{u\in A|u+e\in E$ for all $e\in E\}$ and $V=\{v\in B|v+f\in F$ for all $f\in F\}$. (Thus U and V are the unions of those Archimedean components of A and B lying above E and F in the maximal semilattice quotients of A and B.) It is easy to verify that U and V satisfy the hypotheses of proposition 1 below. Thus $U\otimes V\subset A\otimes B$ and the complement of $U\otimes V$ is an ideal of $A\otimes B$.

H is an ideal of E [1, page 136] and E is clearly an ideal of U. We verify that H is an ideal of U: Let z be the identity element of H. For any $h\in H$ and $u\in U$ we have $h+u = h+z+u$ and since $z+u\in E$ we have $h+u\in H$ as required. Similarly K is an ideal of V. H and K satisfy the hypotheses of proposition 2. Thus $H\otimes K\subset \subset U\otimes V\subset A\otimes B$. At this point we have Fulp's result. Now let S be the maximal subgroup of $A\otimes B$ containing $H\otimes K$. Since the complement of $U\otimes V$ is an ideal we must have $H\otimes K\subset S\subset U\otimes V$. Since $H\otimes K$ is an ideal of $U\otimes V$ we must have $H\otimes K=S$. Thus $H\otimes K$ is a maximal subgroup of $A\otimes B$.

§ 2. Imbedding tools.

Proposition 1. If U is a subsemigroup of A whose complement is an ideal and V is a subsemigroup of B whose complement is an ideal then $U\otimes V$ is imbedded in $A\otimes B$ and its complement is an ideal.

PROOF. Let i, j be the inclusion maps of H and K into U and V. Let C and D be the complements of U and V. Let $(U\otimes V)^t$ be the semigroup whose elements are those of $U\otimes V$ together with an element t (assume $t\notin U\otimes V$) where addition in $U\otimes V$ is as usual and where any sum in which t appears has the value t. The function $\beta:A/C\times B/D\to(U\otimes V)^t$ defined by $(u, v)\beta=u\otimes v$, $(u, D)\beta=t$, $(C, v)\beta=t$, and $(C, D)\beta=t$ is biadditive. Thus there is a homomorphism $\gamma:A/C\otimes B/D\to \to(U\otimes V)^t$ for which $(u\otimes v)\gamma=u\otimes v$ for all $u\in U$, $v\in V$. The composite homomorphism $U\otimes V\xrightarrow{i\otimes j} A\otimes B\to A/C\otimes B/D\xrightarrow{\gamma}(U\otimes V)^t$, where the middle map is induced by the two natural maps, is 1—1. Consequently $i\otimes j$ is 1—1 and $U\otimes V$ is imbedded in $A\otimes B$. The complement of $U\otimes V$ in $A\otimes B$ is the complete inverse image of the ideal $\{t\}$ of $(U\otimes V)^t$ and is therefore an ideal of $A\otimes B$.

Proposition 2. If H is both a subgroup and an ideal of U and K is both a subgroup and an ideal of V then $H\otimes K$ is imbedded in $U\otimes V$ and is both a subgroup and an ideal of $U\otimes V$.

PROOF. Let i, j be the inclusion maps of H and K into U and V. Let e and f be the identity elements of H and K. For the homomorphisms $\varrho:U\to H$ and $\sigma:V\to K$ defined by $u\varrho = u+e$ and $v\sigma = v+f$, the composites $i\varrho$ and $j\sigma$ are identity maps. Then $(i\otimes j)\circ(\varrho\otimes\sigma)$ is the identity map on $H\otimes K$ and consequently $i\otimes j$ is 1—1, i.e. $H\otimes K\subset U\otimes V$. Since $H\otimes K$ is a group we have only to verify that $H\otimes K$ is an ideal. For this purpose it is sufficient to prove that $h\otimes k+u\otimes v$ is in $H\otimes K$ for all $h\in H, k\in K, u\in U, v\in V$. Since H is both an ideal and a subgroup, there is an $h'\in H$ for which $h = h'+u$. Since K is an ideal we have $k' = k+v\in K$. Since K is a subgroup, there is a $k''\in K$ for which $k = k'+k''$. We compute: $h\otimes k+u\otimes v =$

$= (h'+u)\otimes k+u\otimes v = h'\otimes k+u\otimes(k+v) = h'\otimes k+u\otimes k' = h'\otimes(k'+k'')+ +u\otimes k' = (h'+u)\otimes k'+h'\otimes k'' = h\otimes k'+h'\otimes k''$ which is in $H\otimes K$.

Proposition 2 is essentially identical with theorem 1 of [5] and has been included here for ease of reference.

Added in proof: Since this paper was written, P. A. GRILLET has independently initiated the study of tensor products of semigroups in two papers that appeared in *Transactions of the American Mathematical Society*, Vol. **138**, April, 1969: *The tensor product of semigroups* (pages 267—280), and *The tensor product of commutative semigroups* (pages 281—293). The content of our proposition 1 is contained in theorem 4.1 of the first of Grillet's papers referred to above.

References

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