

Stability of the Cauchy equation on a restricted domain

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Abstract. Let G be a commutative semigroup, E a Banach space and $D \subset G \times G$. Let $\varepsilon > 0$ be given and let $f : G \rightarrow E$ satisfy the inequality

$$\|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq \varepsilon \text{ for } (\alpha, \beta) \in D.$$

We prove that under certain assumptions on D and G there exists a unique additive function $A : G \rightarrow E$ such that

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in G.$$

By $\mathbb{R}_+$ we understand the set $[0, \infty)$, by $\mathbb{N}$ the set of all positive integers and by $\mathcal{P}(X)$ the family of all subsets of X . By a vector space we mean a real vector space. Let G be a semigroup. According to the tradition, we will use the additive notation (even in the case when G is noncommutative). When G is a semigroup without zero $G \setminus \{0\}$ denotes G . If $D \subset G \times G$ then we write

$$D_X = \{x | (x, y) \in D\}, D_Y = \{y | (x, y) \in D\}, D_{X+Y} = \{x+y | (x, y) \in D\}.$$

The set D is called a Pexider domain of stability if there exists a constant $K > 0$ such that whenever three functions $f : D_X \rightarrow \mathbb{R}$, $g : D_Y \rightarrow \mathbb{R}$, $h : D_{X+Y} \rightarrow \mathbb{R}$ satisfy the inequality

$$|f(x) + g(y) - h(x + y)| \leq \varepsilon \text{ for } (x, y) \in D,$$

then there exists an additive function $A : G \rightarrow \mathbb{R}$ and two constants $a, b \in \mathbb{R}$ such that

$$\begin{aligned} |f(x) - A(x) - a| &\leq K\varepsilon \text{ for } x \in D_X, \\ |g(y) - A(y) - b| &\leq K\varepsilon \text{ for } y \in D_Y, \\ |h(z) - A(z) - a - b| &\leq K\varepsilon \text{ for } z \in D_{X+Y}. \end{aligned}$$

To our knowledge this definition was introduced by Zs. PÁLES.

We say that a set D is a Cauchy domain of stability if there exists a constant $K > 0$ such that whenever $f : D_X \cup D_Y \cup D_{X+Y} \rightarrow \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon \text{ for } (x, y) \in D,$$

then there exists an additive function $A : G \rightarrow \mathbb{R}$ such that

$$|f(x) - A(x)| \leq K\varepsilon \text{ for all } x \in D_X \cup D_Y \cup D_{X+Y}.$$

On the 32-nd International Symposium on Functional Equations Zs. PÁLES posed the following problem [4]: Is the set $D = \{(x, y) : y \geq x^2\}$ a Pexider domain of stability? The answer to this question is negative (c.f. [6]). We take $f(x) = 0$, $g(x) = h(x) = \ln(1 + |x|)$.

One may ask a closely related question. Is the same set a Cauchy domain of stability? We will show (see Example 1) that the answer is positive. This paper was inspired by this problem.

Definition 1. Let G be a commutative semigroup and let $B \subset G \times G$. We say that $W \subset G$ is B -bounded if

- (i) $\forall \alpha \in G \setminus \{0\} \exists k_0 \in \mathbb{N} \forall k \geq k_0 : k\alpha \notin W$,
- (ii) $\forall \alpha, \beta \in G \setminus \{0\}, (\alpha, \beta) \in B \exists k_0 \in \mathbb{N} \forall k_1 \geq k_0, k_2 \geq 1 : k_1\alpha + k_2\beta \notin W$.

One can easily notice that the family of B -bounded subsets of a semigroup forms an ideal of sets.

Definition 2. Let G be a semigroup. We say that $B \subset G \times G$ is full in G if for every group H , and every function $A : G \rightarrow H$ such that

$$(1) \quad A(\alpha + \beta) = A(\alpha) + A(\beta) \text{ for } (\alpha, \beta) \in B,$$

A is additive.

For a broader study and the literature concerning full sets see [1], [2] or [3].

Proposition 1. *Let G be a semigroup and let $B \subset G \times G$ satisfy the following condition:*

$$(2) \quad \forall (\alpha, \beta) \in (G \times G) \setminus B \exists \gamma \in G : (\beta, \gamma), (\alpha, \beta + \gamma), (\alpha + \beta, \gamma) \in B.$$

Then B is full in G .

PROOF. Consider an arbitrary group H and an arbitrary mapping $A : G \rightarrow H$ satisfying (1). We have to show that A is additive. Let $(\alpha, \beta) \in (G \times G) \setminus B$. Then by (2) there exists a $\gamma \in G$ such that

$$(\beta, \gamma), (\alpha, \beta + \gamma), (\alpha + \beta, \gamma) \in B,$$

so

$$\begin{aligned} A(\beta + \gamma) &= A(\beta) + A(\gamma), \\ A(\alpha + \beta + \gamma) &= A(\alpha) + A(\beta + \gamma), \\ A(\alpha + \beta + \gamma) &= A(\alpha + \beta) + A(\gamma). \end{aligned}$$

This implies that

$$A(\alpha + \beta) + A(\gamma) = A(\alpha + \beta + \gamma) = A(\alpha) + A(\beta + \gamma) = A(\alpha) + A(\beta) + A(\gamma).$$

As H is a group we obtain that $A(\alpha + \beta) = A(\alpha) + A(\beta)$. $\square$

Definition 3. Let E be a vector space, and let $S \subset E$. We define

$$B(S) := \{(x, y) \in S \times S : y \neq rx \text{ for all } r \in \mathbb{R}_-\}.$$

Proposition 2. *Let E be a vector space such that $\dim E \geq 2$. Then $B(E)$ is full.*

PROOF. We are going to show that (2) holds. Let $(x, y) \in E \times E \setminus B(E)$. Then $y = rx$ for a certain $r \in \mathbb{R}_-$. Since $\dim E \geq 2$ we can find a $z \in E$, such that $z \neq rx$ for every $r \in \mathbb{R}$. Then obviously $(y, z) \in B(E)$, $(x, y + z) \in B(E)$, $(x + y, z) \in B(E)$. Proposition 1 completes the proof. $\square$

Let E be a vector space, and let $W \subset E$. One can easily notice that if the intersection of W with any two-dimensional subspace P of E is bounded in P then W is $B(E)$ -bounded. This implies that every bounded subset of a topological vector space is $B(E)$ -bounded.

Suppose that any three elements of W are linearly independent over the field $\mathbb{Q}$. Then W is $B(E)$ -bounded. Condition (i) of the Definition 1 is obvious. Suppose that condition (ii) does not hold. Then we can find $x, y \in$

$E \setminus \{0\}$, $(x, y) \in B(E)$ and $\{k_i\}, \{l_i\} \subset \mathbb{N}$, $\{k_i\}$ an increasing sequence, such that

$$z_i = k_i x + l_i y \in W \text{ for } i \in \mathbb{N}.$$

Since $x \neq 0$ and $\{k_i\}$ is increasing, we can find n_1, n_2, n_3 such that $z_{n_1} \neq z_{n_2}$, $z_{n_2} \neq z_{n_3}$, $z_{n_1} \neq z_{n_3}$. But obviously $z_{n_1}, z_{n_2}, z_{n_3}$ are linearly dependent over $\mathbb{Q}$. We obtain a contradiction.

The above observations and the fact that the finite union of $B(E)$ -bounded sets is still a $B(E)$ -bounded set shows that the family of $B(E)$ -bounded subsets of a vector space is quite large.

Theorem 1. *Let G be a nontrivial commutative semigroup and let $B \subset G \times G$ be full in G . Let $W : G \rightarrow \mathcal{P}(G)$ be a mapping such that $G \setminus W(\alpha)$ is a B -bounded set for every $\alpha \in G$. Let E be a Banach space and let $\varepsilon > 0$. Suppose that $f : G \rightarrow E$ satisfies the following inequality:*

$$\|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq \varepsilon \text{ for } \alpha \in G, \beta \in W(\alpha).$$

Then there exists a unique additive function $A : G \rightarrow E$ such that

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in G.$$

PROOF. Let $\alpha \in G \setminus \{0\}$. Since $G \setminus W(\alpha)$ is B -bounded, there exists $n \in \mathbb{N}$ such that $i\alpha \in W(\alpha)$ for $i \geq n$. Then we have for $k > n$

$$\begin{aligned} \left\| \frac{f(k\alpha)}{k} - f(\alpha) \right\| &\leq \left\| \frac{f(k\alpha) - (k-n)f(\alpha)}{k} \right\| + \left\| \frac{nf(\alpha)}{k} \right\| \\ &\leq \sum_{i=n}^{k-1} \frac{\|f((i+1)\alpha) - f(i\alpha) - f(\alpha)\|}{k} + \frac{1}{k} \|f(n\alpha)\| + \frac{1}{k} \|nf(\alpha)\| \\ &\leq \frac{k-n}{k} \varepsilon + \frac{1}{k} \|f(n\alpha)\| + \frac{1}{k} \|mnf(\alpha)\|. \end{aligned}$$

Thus

$$(3) \quad \limsup_{k \rightarrow \infty} \left\| \frac{f(k\alpha)}{k} - f(\alpha) \right\| \leq \varepsilon \text{ for } \alpha \in G \setminus \{0\}.$$

Replacing in this inequality α by $n\alpha$ and dividing both sides by n we obtain

$$\limsup_{k \rightarrow \infty} \left\| \frac{f(kn\alpha)}{kn} - \frac{f(n\alpha)}{n} \right\| \leq \frac{\varepsilon}{n},$$

and inserting l instead of k and m instead of n

$$\limsup_{l \rightarrow \infty} \left\| \frac{f(ml\alpha)}{ml} - \frac{f(m\alpha)}{m} \right\| \leq \frac{\varepsilon}{m}.$$

Thus

$$\left\| \frac{f(n\alpha)}{n} - \frac{f(m\alpha)}{m} \right\| \leq \frac{\varepsilon}{n} + \frac{\varepsilon}{m}.$$

This proves that $\left\{ \frac{f(n\alpha)}{n} \right\}$ is a Cauchy sequence for $\alpha \in G \setminus \{0\}$. If G is a semigroup with zero then obviously $\left\{ \frac{f(n0)}{n} \right\}$ is convergent. We define

$$A(\alpha) := \lim_{n \rightarrow \infty} \frac{f(n\alpha)}{n} \text{ for } \alpha \in G.$$

Due to (3) and the definition of A we have $\|f(\alpha) - A(\alpha)\| \leq \varepsilon$ for $\alpha \in G \setminus \{0\}$. In the case when G is a semigroup with zero there exists a $\beta \notin W(0)$. Then we have $\|f(0 + \beta) - f(0) - f(\beta)\| \leq \varepsilon$, so $\|f(0)\| \leq \varepsilon$. Hence

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in G,$$

and replacing α by $n\alpha$ and dividing by n we obtain

$$\left\| \frac{f(n\alpha)}{n} - A(\alpha) \right\| \leq \frac{\varepsilon}{n} \text{ for } \alpha \in G.$$

We prove the additivity of A . Let $\alpha, \beta \in G$. Suppose that $(\alpha, \beta) \in B$. As $A(0) = 0$ we may assume that $\alpha, \beta \in G \setminus \{0\}$. Consider an arbitrary $\delta > 0$ and take an $n \in \mathbb{N}$ such that $\frac{\varepsilon}{n} \leq \delta$. Then, as $G \setminus W(n\beta)$ is B -bounded, there exists $k \in \mathbb{N} \setminus \{0\}$ such that

$$i\alpha + j(n\beta) \in W(n\beta) \text{ for } i \geq nk, j \geq 1.$$

Hence

$$\begin{aligned} \|A(\alpha + \beta) - A(\alpha) - A(\beta)\| &\leq \left\| \frac{f(nk(\alpha + \beta))}{nk} - \frac{f(nk\alpha)}{nk} - \frac{f(n\beta)}{n} \right\| + 3\delta \\ &\leq \sum_{i=0}^{k-1} \frac{1}{nk} \|f((i+1)n\beta + nk\alpha) - f(in\beta + nk\alpha) - f(n\beta)\| + 3\delta \\ &\leq \frac{k}{nk} \varepsilon + 3\delta \leq 4\delta. \end{aligned}$$

Since δ was chosen arbitrarily, we obtain that $A(\alpha + \beta) = A(\alpha) + A(\beta)$. As B is full, this proves that A is additive. Because $A(\alpha) = \lim_{n \rightarrow \infty} \frac{f(n\alpha)}{n}$, A is a unique additive approximation of f . $\square$

Corollary 1. *Let G be a semigroup, and let H be a nontrivial subsemigroup of the centre of G . Let $B \subset H \times H$ be full in H . Let $W : G \rightarrow \mathcal{P}(H)$ be a mapping such that $H \setminus W(\alpha)$ is B -bounded for $\alpha \in H$. We assume the following condition:*

(4) *for every $\alpha \in G$ there exists a $\beta \in W(\alpha)$ such that $\alpha + \beta \in H$.*

Let F be a Banach space, and let $\varepsilon > 0$. Suppose that $f : G \rightarrow F$ satisfies the inequality

$$\|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq \varepsilon \text{ for } \alpha \in G, \beta \in W(\alpha).$$

Then there exists a unique additive function $A : G \rightarrow F$ such that

$$\begin{aligned} \|f(\alpha) - A(\alpha)\| &\leq 3\varepsilon \text{ for } \alpha \in G, \\ \|f(\alpha) - A(\alpha)\| &\leq \varepsilon \text{ for } \alpha \in H. \end{aligned}$$

PROOF. Making use of Theorem 1 for the function $f|_H$ we obtain that there exists a unique additive function $A : H \rightarrow F$ such that

$$\|f(\alpha) - A(\alpha)\| \leq \varepsilon \text{ for } \alpha \in H.$$

We will show that A has a unique additive extension onto G . Let $\alpha \in G$. Then by (4) there exists a $\beta \in W(\alpha) \subset H$ such that $\alpha + \beta \in H$. We define $\tilde{A}(\alpha) := A(\alpha + \beta) - A(\beta)$. Now we prove that $\tilde{A}$ is well defined. Suppose that $\alpha + \beta_1, \alpha + \beta_2 \in H$ for certain $\beta_1, \beta_2 \in H$. Then

$$\begin{aligned} (A(\alpha + \beta_1) - A(\beta_1)) - (A(\alpha + \beta_2) - A(\beta_2)) \\ = A(\alpha + \beta_1 + \beta_2) - A(\alpha + \beta_2 + \beta_1) = 0. \end{aligned}$$

Making use of the fact that H is contained in the center of G one can easily prove that $\tilde{A}$ is additive. The way of defining $\tilde{A}$ shows that it is a unique additive extension of A .

Let $\alpha \in G$. Then by (4) there exists a $\beta \in W(\alpha)$ such that $\alpha + \beta \in H$. Then

$$\begin{aligned} \|f(\alpha) - \tilde{A}(\alpha)\| &\leq \|f(\alpha + \beta) - A(\alpha + \beta) + A(\beta) - f(\beta)\| \\ &\quad + \|f(\alpha + \beta) - f(\alpha) - f(\beta)\| \leq 3\varepsilon. \end{aligned} \quad \square$$

Corollary 2. *Let E be a vector space, let $C \subset E$ be a convex cone such that $C \cap -C = \{0\}$, $C - C = E$. Let $V : E \rightarrow \mathcal{P}(E)$ be a mapping such that $C \setminus V(x)$ is a $B(E)$ -bounded set for $x \in E$. Let F be a Banach space and let $\varepsilon > 0$. Suppose that $f : E \rightarrow F$ satisfies the inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon \text{ for } x \in E, y \in W(x).$$

Then there exists a unique additive function $A : E \rightarrow F$ such that

$$\|f(x) - A(x)\| \leq 3\varepsilon \text{ for } x \in E,$$

$$\|f(x) - A(x)\| \leq \varepsilon \text{ for } x \in C.$$

PROOF. We are going to show that the assumptions of Corollary 1 are satisfied. Let $W(x) := C \cap V(x)$. Because C is a cone such that $C \cap -C = \{0\}$, $B := B(E) \cap (C \times C) = C \times (C \setminus \{0\})$. Hence B is full in C and $C \setminus W(x)$ is B -bounded for $x \in C$. Let $x \in E$. Because $C - C = E$, there exists an $a \in C$ such that $a + x \in C$. Since $C \setminus W(x)$ is $B(E)$ -bounded there exists a $b \in C$ such that $a + b \in W(x)$. Then obviously $(a + b) + x \in C$. Corollary 1 completes the proof. $\square$

Example 1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$. We put $E = F = \mathbb{R}$, $C = \mathbb{R}_+$, $V(x) = (g(x), +\infty)$. Now by Corollary 2 we obtain that the set

$$D = \{(x, y) : y > g(x)\}$$

is a Cauchy domain of stability.

Corollary 3. *Let E be a vector space, let F be a Banach space and let $\varepsilon > 0$. Suppose that $W : E \rightarrow \mathcal{P}(E)$ is a mapping, such that $E \setminus W(x)$ is a $B(E)$ -bounded set for $x \in E$. Suppose that $f : E \rightarrow F$ satisfies the following inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon \text{ for } x \in E, y \in W(x).$$

Then there exists a unique additive function $A : E \rightarrow F$ such that

$$\|f(x) - A(x)\| \leq \varepsilon \text{ for } x \in E.$$

PROOF. Suppose that $E = \mathbb{R}$. Let $g(x) = -f(-x)$. Then

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \varepsilon \text{ for } y \in W(x), \\ \|g(x + y) - g(x) - g(y)\| &\leq \varepsilon \text{ for } y \in -W(-x). \end{aligned}$$

Obviously the set $-W(-x)$ is $B(\mathbb{R})$ -bounded for every $x \in R$. Now due to Corollary 2 we can find additive functions $A_1, A_2 : E \rightarrow F$ such that

$$\begin{aligned}\|f(x) - A_1(x)\| &\leq 3\varepsilon \text{ for } x \in \mathbb{R}, \\ \|f(x) - A_1(x)\| &\leq \varepsilon \text{ for } x \in \mathbb{R}_+, \\ \|g(x) - A_2(x)\| &\leq 3\varepsilon \text{ for } x \in \mathbb{R}, \\ \|g(x) - A_2(x)\| &\leq \varepsilon \text{ for } x \in \mathbb{R}_+.\end{aligned}$$

One can easily notice that then $A_1 = A_2 =: A$. Hence

$$\begin{aligned}\|f(x) - A(x)\| &\leq \varepsilon \text{ for } x \in \mathbb{R}_+, \\ \|-f(-x) - A(x)\| &\leq \varepsilon \text{ for } x \in \mathbb{R}_+.\end{aligned}$$

The last two inequalities mean that

$$\|f(x) - A(x)\| \leq \varepsilon \text{ for } x \in \mathbb{R}.$$

Suppose that $\dim E \geq 2$. Then by Proposition 2 $B(E)$ is full, so Theorem 1 completes the proof. $\square$

Putting $E = \mathbb{R}$, $W(x) := [-1, 1]$ we obtain a generalization of Theorem 3 [5]. Moreover we get the best possible constant K (instead of $K = 9$ we have $K = 1$).

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