

On bi-ideals and quasi-ideals in semigroups

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The purpose of this paper is twofold: an examination of elementary properties of bi-ideals and an investigation of the relationships between bi-ideals and quasi-ideals. In the first section we define an equivalence relation $\mathcal{B}$ on a semigroup which is finer than $\mathcal{H}$ and show that any bi-ideal is the union of $\mathcal{B}$ -classes. An investigation of O -minimal bi-ideals follows. We show that in a semigroup with O , a bi-ideal is O -minimal if and only if it is a non-zero $\mathcal{B}$ -class union $\{O\}$.

In the second section we show that under certain conditions a O -minimal bi-ideal is also a O -minimal quasi-ideal; this is not always true. We say that a semigroup is in the class $\mathcal{B}\mathcal{Q}$ whenever its sets of bi-ideals and quasi-ideals coincide. Several sufficient conditions are found (eg. right O -simple, regular) for a semigroup to be in this class. An example is given which sheds some further light on the characterization of $\mathcal{B}\mathcal{Q}$ semigroups. We conclude with a characterization of these semigroups announced by Calais at the Semigroup Symposium in Bratislava, Czechoslovakia in June, 1968.

We follow the notation and terminology of [3]. We will always use $\subset$ for proper containment. Equivalence relations will be denoted by script letters with the subscripted capital italic denoting the corresponding equivalence class. Thus R_a denotes the $\mathcal{R}$ -class of a .

1. Bi-ideals

(1.1) *Definition.* A (non-empty) subset B of a semigroup S is a bi-ideal if $BS^1B \subseteq B$. (Clearly a bi-ideal is a subsemigroup.)

We now define a relation on a semigroup which will be useful in our investigation of bi-ideals.

(1.2) *Definition.* For $a, b \in S$, a given semigroup, we write $a\mathcal{B}b$ if 1) $a=b$ or 2) there exists $u, v \in S$ such that $aua=b$ and $bvb=a$.

The following two propositions can be readily verified:

(1.3) *Proposition.* The relation $\mathcal{B}$ defined in (1.2) is an equivalence relation; indeed $\mathcal{B} \subseteq \mathcal{H}$.

(1.4) *Proposition.* If A is a bi-ideal of a semigroup then $A = \bigcup_{a \in A} B_a$, i.e., any bi-ideal is the union of its $\mathcal{B}$ -classes.

In what follows we will be concerned mainly with semigroups with O though we could proceed as in [8] for semigroups with proper Suschkewitsch kernel and obtain more general results. With this in mind we recall the following:

(1.5) *Definition.* A non-zero bi-ideal B of a semigroup S with O is said to be O -minimal if there is no bi-ideal B' of S with $\{O\} \subset B' \subset B$.

The following result follows immediately from (1.4):

(1.6) *Corollary.* Let S be a semigroup with O . If a bi-ideal, B , is a non-zero $\mathcal{B}$ -class union $\{O\}$ then it is a O -minimal bi-ideal.

The converse of this corollary is also true as we show in the following:

(1.7) *Theorem.* Let S be a semigroup with O . A bi-ideal is O -minimal if and only if it is a non-zero $\mathcal{B}$ -class union $\{O\}$.

PROOF. Let B be a O -minimal bi-ideal of S . Let $a, b \in B \setminus \{O\}$. Since $\{b, b^2\} \cup bSb$ and $\{a, a^2\} \cup aSa$ are clearly non-zero bi-ideals contained in B we must have $B = \{b, b^2\} \cup bSb = \{a, a^2\} \cup aSa$.

Now assume $a \neq b$. We can proceed from the last equality by cases.

Suppose $a = b^2$. We have two sub-cases to consider.

1) If also $b = a^2$ then $a = b^2 = aa^2a = b(ba^2b)b$ and also $b = a^2 = a(ab^2a)a$. It follows that $a\mathcal{B}b$.

2) If $b \neq a^2$ we must have $b \in aSa$ and $b = aua$ for some $u \in S$. Then $a = b^2 = auaaua = b(buaaub)b$. Again it follows that $a\mathcal{B}b$.

Now if $a \neq b$ and $a \neq b^2$ we must have $a \in bSb$ so that $a = bvb$ for some $v \in S$. Again we examine b by cases as above. If $b = a^2$ we have simply case 2) with the roles of a and b interchanged. If $b \in aSa$ then $b = aua$ for some $u \in S$. In either case it follows that $a\mathcal{B}b$.

By (1.4) we may conclude that $B = B_b \cup \{O\}$.

The converse is just (1.6).

We now remark, with thanks to Professor Otto Steinfield and Mr. Bruce Mielke, that the above proof suffices to show that $a\mathcal{B}b$ if and only if $B(a) = \{a, a^2\} \cup aSa = \{b, b^2\} \cup bSb = B(b)$, i.e., two elements are $\mathcal{B}$ related precisely when their principal bi-ideals coincide. Thus $\mathcal{B}$ has the same relation to bi-ideals as $\mathcal{L}$ to left ideals, $\mathcal{R}$ to right ideals, $\mathcal{H}$ to quasi-ideals [4] and $\mathcal{I}$ to two-sided ideals.

We now proceed to investigate the structure of O -minimal bi-ideals. The reader will recall that a null subsemigroup N is a subsemigroup with O in which $ab = O$ for any $a, b \in N$.

(1.8) *Theorem.* Let S be a semigroup with O . A O -minimal bi-ideal, B , of S is either a null subsemigroup or a group with $\{O\}$.

PROOF. From (1.7) we have $B = B_b \cup \{O\}$ for any $b \in B \setminus \{O\}$. We recall from (1.3) that $\mathcal{B} \subseteq \mathcal{H}$. Thus if $b^2 \neq O$ we conclude $b^2\mathcal{B}b$ and $b^2\mathcal{H}b$. It now follows that H_b is a group ([3] Theorem 2.16). Now if $a \in H_b$ we would also have $a\mathcal{B}b$ since the equations $a = bvb$ and $aua = b$ can be solved for u and v in the group H_b . Thus B is the group H_b union $\{O\}$.

On the other hand if $b^2 = O$ for each $b \in B \setminus \{O\}$ and if $a \in B \setminus \{O\}$ we have $a\mathcal{B}b$, $a\mathcal{H}b$ and then $ab\mathcal{D}b^2 = O$ by [3] Theorem 2.4. Since it is clear that $D_O = \{O\}$ we have $ab = O$ and B is a null subsemigroup.

We now proceed to investigate the relationship between 0-minimal left and right ideals and 0-minimal bi-ideals.

(1.9) Proposition. *Let S be a semigroup with 0. If R is a 0-minimal right ideal and L is a 0-minimal left ideal then either $RL = \{0\}$ or RL is a 0-minimal bi-ideal of S .*

PROOF. Suppose $RL \neq \{0\}$ and that there is a bi-ideal B with $\{0\} \subset B \subset RL$. Since $RL \subseteq R \cap L$ we have $BS^1 \subseteq RS^1 \subseteq R$. It follows that $RS^1 = R$. Now $B \subseteq L$ so that we have $B \subset RL = BS^1B \subseteq B$, a contradiction. Since RL is a bi-ideal ([3] p 85 ex 18 c) the proof is complete.

Unfortunately not every 0-minimal bi-ideal in a semigroup with 0 can be obtained as the product of a 0-minimal right ideal and a 0-minimal left ideal as the following simple example shows.

(1.10) Example. Let S be the semigroup $\{a, a^2, a^3, 0\}$ where $a^4 = 0$. It is easy to check that $\{a^2, 0\}$ is a 0-minimal bi-ideal which is not the product of a 0-minimal right ideal and 0-minimal left ideal.

This also fails even when the 0-minimal bi-ideal is a group with $\{0\}$ as in the following:

(1.11) Example. Let S be the semigroup consisting of those 2×2 matrices of the form $\begin{pmatrix} i & 0 \\ j & a \end{pmatrix}$ where i, j are non-negative integers and $a = 0, 1$. One readily checks that $B = \left\{0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a 0-minimal bi-ideal; indeed B is even a 0-minimal left ideal. $\left(0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$. Moreover, B is a group with $\{0\}$. Since there are no 0-minimal right ideals in S we can not obtain B as the product of a 0-minimal right ideal and 0-minimal left ideal. However we do have the following:

(1.12) Proposition. *Let S be a semigroup with 0. If B is a 0-minimal bi-ideal of S then for any right ideal R contained in BS^1 and any left ideal L contained in S^1B we have either $RL = \{0\}$ or $RL = B$.*

PROOF. Let $R \subseteq BS^1$ and $L \subseteq S^1B$. Then $RL \subseteq BS^1S^1B \subseteq BS^1B \subseteq B$. Since RL is a bi-ideal and B is 0-minimal it follows that $RL = \{0\}$ or $RL = B$.

We note that the semigroup in example (1.11) shows that RL may be $\{0\}$ even if B is a group with $\{0\}$ if we take $R = \left\{\begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} \mid j \geq 0\right\}$ and $L = B$. (However $SB = B$.)

We now conclude this section with the following proposition which we are informed can also be found in [5]:

(1.13) Proposition. *The product of two bi-ideals in a semigroup is always a bi-ideal.*

PROOF. Let A and B be two bi-ideals of a given semigroup S . Then $(AB)S^1(AB) = A(B(S^1A)B) \subseteq A(BSB) \subseteq AB$. It follows that AB is a bi-ideal.

2. Bi-ideals and quasi-ideals

In this section we examine the relationship between bi-ideals and quasi-ideals. The desirable property of bi-ideals proven in (1. 13) and Steinfeld's question [7] as to whether quasi-ideals also had this property motivated much of the following investigation. The reader will recall the following definition and proposition.

(2. 1) *Definition.* (A non-empty) subset Q of a semigroup S is called a quasi-ideal if $QS \cap SQ \subseteq Q$.

(2. 2) *Proposition.* ([3] p 85 ex 18). *Every quasi-ideal of a semigroup is a bi-ideal. In a regular semigroup every bi-ideal is also a quasi-ideal.*

It is now natural to define the following class of semigroups. It is clearly non-vacuous by (2. 2):

(2. 3) *Definition.* The class $\mathcal{B}\mathcal{Q}$ of semigroups will consist precisely of those semigroups whose sets of bi-ideals and quasi-ideals coincide.

Because of (1. 13) the following corollary is immediate:

(2. 4) *Corollary.* The product of two quasi-ideals of a semigroup $S \in \mathcal{B}\mathcal{Q}$ is always a quasi-ideal.

We now proceed in two directions in an attempt to find the extent of the class $\mathcal{B}\mathcal{Q}$. First we try to determine large subclasses of $\mathcal{B}\mathcal{Q}$, then we will try to determine the relationship between the minimal bi-ideals and quasi-ideals in a semigroup and to see whether this has any determinable effect on the total sets of bi-ideals and quasi-ideals of the semigroup. Preliminary to this we quote the following theorem of Steinfeld:

(2. 5) *Theorem* ([7] Theorem 1). The intersection of a left and right ideal of a semigroup is a quasi-ideal. Conversely, every quasi-ideal of S can be obtained as the intersection of a left and right ideal.

(2. 6) *Proposition.* *If S is a left [right] simple semigroup then each bi-ideal of S is a right [left] ideal.*

PROOF. Suppose S is left simple and let B be a bi-ideal of S . Since S^1B is a left ideal of S and S is left simple we must have $S^1B = S$. Thus $B \supseteq BS^1B = B(S^1B) = BS$ and it follows that B is a right ideal. The proof for a right simple semigroup is dual.

The following corollary is now an immediate result of the above and (2. 5):

(2. 7) *Corollary.* If S is a left [right] simple semigroup then $S \in \mathcal{B}\mathcal{Q}$.

We can modify the above argument to prove:

(2. 8) *Proposition.* *Let S be a semigroup with 0. If S is a left [right] 0-simple semigroup then $S \in \mathcal{B}\mathcal{Q}$.*

PROOF. If B is a non-zero bi-ideal of S then $0 \in BSB \subseteq B$. Thus if $SB = \{0\}$ we would be done since in this case B is a left ideal; otherwise $SB = S$, since S is by hypothesis left 0-simple. We can now proceed as in (2. 6) and (2. 7).

We modify below the semigroup in example (1. 11) to obtain a semigroup which is not in the class $\mathcal{B}\mathcal{Q}$. Indeed this is a rather elementary semigroup in

comparison to the construction in [1] and we also find a right ideal, R , and left ideal L , whose product, RL , is not a quasi-ideal. Since right and left ideals are quasi-ideals the product of two quasi-ideals is not always a quasi-ideal.

(2.9) *Example.* Let S be the multiplicative semigroup of 2×2 matrices, of the form $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ where a and b are positive real numbers. Let $R = \left\{ \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \mid a < b \right\}$ and $L = \left\{ \begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix} \mid q > 5 \right\}$. One can readily check that R is a right ideal and L a left ideal. Moreover, we can directly verify from the defining conditions of R and L that $\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} \notin RL$. But we have

$$\begin{pmatrix} 5 & 0 \\ 10 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \left[\begin{pmatrix} 1/2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 6 & 1 \end{pmatrix} \right] \begin{pmatrix} 1/4 & 0 \\ 1 & 1 \end{pmatrix}$$

so that directly $S[RL] \cap [RL]S \not\subseteq RL$ and RL is not a quasi-ideal.

Now in example (1.10) we can see that $B = \{a^2, 0\}$ is a 0-minimal bi-ideal which is not a quasi-ideal since $a^3 \in BS \cap SB$ but $a^3 \notin B$. However, as the following proposition shows, it is not necessary for a semigroup, S , to be in the class $\mathcal{B}\mathcal{Q}$ in order for the product of two quasi-ideals of S to be a quasi-ideal.

(2.10) *Proposition.* In a commutative semigroup the product of two quasi-ideals is a quasi-ideal.

PROOF. In commutative semigroups the set of quasi-ideals and ideals coincide. The result is thus immediate.

Thus we have seen that if a semigroup is either regular, right 0-simple, or left 0-simple it belongs to the class $\mathcal{B}\mathcal{Q}$.

We now give one further result which is a partial characterization for the class of quasi-regular semigroups. This class properly contains the class of regular semigroups. From Calais [1] we have the following definition and result:

(2.11) *Definition.* A semigroup S is said to be quasi-regular if each right ideal and each left ideal is idempotent.

(2.12) **Theorem.** ([1] Proposition 2.2). A semigroup S is quasi-regular if and only if $Q = (QS)^2 \cap (SQ)^2$ for each quasi-ideal Q of S .

We can now characterize those quasi-regular semigroups which belong to $\mathcal{B}\mathcal{Q}$. We note that Calais' example shows that quasi-regular semi-groups do not in general belong to the class $\mathcal{B}\mathcal{Q}$.

(2.13) **Theorem.** A semigroup S belongs to the class $\mathcal{B}\mathcal{Q}$ and is quasi-regular if and only if $B = (BS)^2 \cap (SB)^2$ for each bi-ideal B of S .

PROOF. If we have $B = (BS)^2 \cap (SB)^2$ for each bi-ideal B of S then S is clearly quasi-regular by (2.12) since by (2.2) every quasi-ideal of S is also a bi-ideal. But if S is quasi-regular then we have $(BS)^2 = BS$ and $(SB)^2 = SB$ since BS and SB are right and left ideals respectively. Whence for any bi-ideal B we have $B = (BS)^2 \cap (SB)^2 = BS \cap SB$ so that B is also a quasi-ideal. It is now immediate that $S \in \mathcal{B}\mathcal{Q}$.

Conversely if S is quasi-regular and $S \in \mathcal{B}\mathcal{Q}$ then each bi-ideal is a quasi-ideal. The result follows immediately upon application of (2. 12).

We now turn to the problem of the relationship of 0-minimal bi-ideals and 0-minimal quasi-ideals. We first give a theorem due to Steinfeld:

(2. 14) **Theorem.** ([8] Theorem 2). *Let S be a semigroup with 0 and let R and L be 0-minimal right and left ideals of S respectively. Then either RL is a null semigroup or it is a group with 0. In the latter case we have $RL = R \cap L$ and thus RL is a 0-minimal quasi-ideal of S .*

From this we may deduce the following:

(2. 15) **Theorem.** *Let S be a semigroup with 0. If B is 0-minimal bi-ideal which is a group with $\{0\}$ and can be written as the product of a 0-minimal right ideal and 0-minimal left ideal then B is also a 0-minimal quasi-ideal.*

Now if we return to the semigroup in example (1. 11) we see that B , a 0-minimal bi-ideal, is a group with $\{0\}$ which is also a 0-minimal quasi-ideal even though it cannot be written as a product of a 0-minimal right ideal and a 0-minimal left ideal. That 0-minimal bi-ideals which are groups with $\{0\}$ are 0-minimal quasi-ideals is shown in a note by Mr. Bruce Mielke submitted to this Journal.

When a semigroup S does not have a 0 or if we restrict ourselves to minimal bi-ideals and quasi-ideals the problem is much simpler. We conclude the paper with a discussion of this case. First from (1. 12) we can deduce:

(2. 16) **Proposition.** *Let S be a semigroup. If B is a minimal bi-ideal of S then BS and SB are minimal right and left ideals of S respectively and we have $B = (BS)(SB)$, i.e. B is the product of a minimal right ideal and a minimal left ideal.*

PROOF. Let R be a right ideal and L a left ideal of S with $R \subseteq BS$, $L \subseteq SB$. Since RL is a bi-ideal with $RL \subseteq BSSB \subseteq BS^1B \subseteq B$ we must have $B = RL$. But $RL \subseteq R \cap L$ and thus $B \subseteq R$. Hence we have $BS \subseteq RS \subseteq R$ so that $R \subset BS$ would lead to a contradiction. It follows that BS is a minimal right ideal. Similarly SB is a minimal left ideal. Since B is a minimal bi-ideal which contains $(BS)(SB)$, itself a bi-ideal, we must have $B = (BS)(SB)$.

(2. 17) **Lemma** ([2] Lemma 3. 4). *If R is a minimal right ideal and L a minimal left ideal of a semigroup then $RL = R \cap L$.*

(2. 18) **Lemma** ([7] Theorem 4a). *Each minimal quasi-ideal of a semigroup is a group.*

(2. 19) **Theorem.** *For any semigroup S the set of minimal bi-ideals and minimal quasi-ideals coincide.*

PROOF. If B is a minimal bi-ideal then $B = RL$ for some minimal right ideal R and minimal left ideal L by (2. 16). Now $RL = R \cap L$ by (2. 17). Thus B is also a quasi-ideal by (2. 5) and it follows from (2. 2) that B is a minimal quasi-ideal.

Conversely let Q be a minimal quasi-ideal. By (2. 18) Q is a group. Now Q is also a bi-ideal and any bi-ideal is the union of $\mathcal{B}$ -classes by (1. 4). Since a group is clearly contained in but one $\mathcal{B}$ -class it follows immediately that Q is a minimal bi-ideal.

Finally we have the characterization given by Calais:

(2. 20) **Theorem** [Calais; Reims, France]. *Let S be a semigroup. Let $B(x, y)$ denote the smallest bi-ideal of S containing $x, y \in S$ and let $Q(x, y)$ denote the smallest quasi-ideal of S containing x, y . Then $S \in \mathcal{BQ}$ if and only if $B(x, y) = Q(x, y)$ for every $x, y \in S$.*

PROOF. The condition $B(x, y) = Q(x, y)$ is clearly necessary.

Now let B be an arbitrary bi-ideal of S and let $z \in BS \cap SB$. Then we have $x, y \in B$ such that $z \in xS \cap Sy$. But $xS \cap Sy \subseteq Q(x, y) = (\{x, y\} \cup \{x, y\}S) \cap (\{x, y\} \cup S\{x, y\})$. Since $Q(x, y) = B(x, y)$ we have $z \in B(x, y)$. Since $B(x, y) = \{x, y\} \cup \{x, y\}^2 \cup \{x, y\}S\{x, y\}$ is the smallest bi-ideal of S containing x, y we can conclude $z \in B(x, y) \subseteq B$. Thus $BS \cap SB \subseteq B$ and B is quasi-ideal. It now follows from (2. 2) that $S \in \mathcal{BQ}$.

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(Received May 13, 1967.)

* The author received partial research support from NSF GP 3993, WARF No. 161—4854 and Graduate School Research Fund 101-8365.