

On a characterization of the Rényi—Shannon entropy for incomplete probability distributions

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RÉNYI [1] has defined axiomatically in the following way the information associated to an incomplete probability distribution: Let π denote the set of all finite sequences $P = (p_1, \dots, p_n)$ of nonnegative numbers such that $0 < w(P) = \sum_{k=1}^n p_k \leq 1$.

If $P \in \pi$ and $Q \in \pi$ where $P = (p_1, \dots, p_n)$, $Q = (q_1, \dots, q_m)$ we put $P * Q = (p_1 q_1, \dots, p_n q_1, \dots, p_1 q_m, \dots, p_n q_m)$. If $P \in \pi$ and $Q \in \pi$ where $P = (p_1, \dots, p_n)$, $Q = (q_1, \dots, q_m)$ and $w(P) + w(Q) \leq 1$ we put $P \cup Q = (p_1, \dots, p_n, q_1, \dots, q_m)$. Let $I(P) = I(p_1, \dots, p_n)$ be defined for $P \in \pi$ such that

A') $I(p)$ is continuous for $0 < p \leq 1$, and $I(1/2) = 1$

B') $I(p_1, \dots, p_n)$ is a symmetric function of its variables,

C') $I(P * Q) = I(P) + I(Q)$, further

D') if $w(P) + w(Q) \leq 1$ then

$$I(P \cup Q) = \frac{w(P)I(P) + w(Q)I(Q)}{w(P) + w(Q)}.$$

Rényi has shown that under the conditions

$$I(P) = \frac{\sum_{k=1}^n p_k \log_2 \frac{1}{p_k}}{\sum_{k=1}^n p_k}.$$

A distribution $P \in \pi$ is called an *incomplete probability distribution*: the function $I(P)$ is called the *information associated to P*.

We want to show here the following.

Theorem. *If for every $P \in \pi$:*

A) $I(p)$ is continuous for $0 < p \leq 1$ and nonnegative,

B) $I(p_1, \dots, p_n)$ is a symmetric function of its variables,

C) $I(p_1, \dots, p_n, 0) = I(p_1, \dots, p_n)$,

D) $I(p_1, \dots, p_n) - I(p_1, \dots, p_{n-2}, p_{n-1} + p_n) = \Phi(p_1 + \dots + p_{n-2}, p_{n-1}, p_n)$
with $\Phi \geq 0$,

$$E) I(P * Q) = I(P) + I(Q),$$

F) $\Phi(0, p, 1-p)$ is a Lebesgue integrable function over the interval $0 \leq p \leq 1$ not identically equal to 0, then we have for some $a > 1$ and $n > 1$

$$I(P) = \frac{\sum_{k=1}^n p_k \log_a \frac{1}{p_k}}{w(P)} + \log_a w(P) - \log_b w(P)$$

In particular if $a = b$, then

$$I(P) = \frac{\sum_{k=1}^n p_k \log_a \frac{1}{p_k}}{\sum_{k=1}^n p_k}.$$

PROOF. From the hypothesis C and D it follows immediately

$$(1) \quad \Phi(p_1 + \dots + p_{n-2}, p_{n-1}, 0) = I(p_1, \dots, p_{n-1}, 0) - I(p_1, \dots, p_{n-1}) = 0.$$

By the symmetry hypothesis, we have

$$I(p_1, \dots, p_{n-1}, p_n) = I(p_1, \dots, p_n, p_{n-1}).$$

From this and from D we obtain

$$(2) \quad \Phi(p_1 + \dots + p_{n-2}, p_{n-1}, p_n) = \Phi(p_1 + \dots + p_{n-2}, p_n, p_{n-1}).$$

By the symmetry hypothesis, we get also:

$$I(p_1, \dots, p_{n-2}, p_{n-1}, p_n) = I(p_1, \dots, p_{n-1}, p_{n-2}, p_n).$$

From this and applying D twice we deduce that necessarily

$$(3) \quad \begin{aligned} &\Phi(p_1 + \dots + p_{n-2}, p_{n-1}, p_n) + \Phi(p_1 + \dots + p_{n-3}, p_{n-2}, p_{n-1} + p_n) = \\ &= \Phi(p_1 + \dots + p_{n-1}, p_{n-2}, p_n) + \Phi(p_1 + \dots + p_{n-3}, p_{n-1}, p_{n-2} + p_n). \end{aligned}$$

From E we deduce, for $m = 1$

$$(4) \quad I(p_1 q, \dots, p_n q) = I(p_1, \dots, p_n) + I(q)$$

and thus in particular, for $n = 1$

$$(4') \quad I(pq) = I(p) + I(q)$$

The relation (4') must be true for all (p, q) with $0 < p \leq 1$, $0 < q \leq 1$; the function $I(p)$ must be continuous and nonnegative for all $p: 0 < p \leq 1$; this implies ([2]):

$$(5) \quad I(p) = c \log p \quad \text{for } 0 < p \leq 1,$$

where c is a negative constant. Applying D many times successively we obtain

$$(6) \quad \begin{aligned} I(p_1, \dots, p_n) &= \Phi(p_1 + \dots + p_{n-2}, p_{n-1}, p_n) + \\ &+ \Phi(p_1 + \dots + p_{n-3}, p_{n-2}, p_{n-1} + p_n) + \dots + \Phi(0, p_1, p_2 + \dots + p_n) + \\ &+ I(p_1 + p_2 + \dots + p_n) \end{aligned}$$

and also, bearing in mind the symmetry of $I(p_1, \dots, p_n)$,

$$\begin{aligned}
 (6') \quad & I(p_1 q_1, \dots, p_n q_1; \dots, p_1 q_m, \dots, p_n q_m) = \\
 & = \Phi[w(P)(q_1 + \dots + q_{m-1}) + (p_1 + \dots + p_{n-2})q_m, p_{n-1}q_m, p_n q_m] + \dots + \\
 & \quad + \Phi[w(P)(q_1 + \dots + q_{m-1}), p_1 q_m, (p_2 + \dots + p_n)q_m] + \dots + \\
 & \quad + \Phi[w(P)(q_2 + \dots + q_m) + (p_1 + \dots + p_{n-2})q_1, p_{n-1}q_1, p_n q_1] + \dots + \\
 & \quad + \Phi[w(P)(q_2 + \dots + q_m), p_1 q_1, (p_2 + \dots + p_n)q_1] + I(w(P)q_1, w(P)q_2, \dots, w(P)q_m).
 \end{aligned}$$

From the relationships (4), (6), (6') it follows from E , that for all $P \in \pi$ and $Q \in \pi$ one has

$$\begin{aligned}
 (7) \quad & \Phi[w(P)(q_1 + \dots + q_{m-1}) + (p_1 + \dots + p_{n-2})q_m, p_{n-1}q_m, p_n q_m] + \dots + \\
 & \quad + \Phi[w(P)(q_1 + \dots + q_{m-1}), p_1 q_m, (p_2 + \dots + p_n)q_m] + \dots + \\
 & \quad + \Phi[w(P)(q_2 + \dots + q_m) + (p_1 + \dots + p_{n-2})q_1, p_{n-1}q_1, p_n q_1] + \dots + \\
 & \quad + \Phi[w(P)(q_2 + \dots + q_m), p_1 q_1, (p_2 + \dots + p_n)q_1] = \\
 & = \Phi[p_1 + \dots + p_{n-2}, p_{n-1}, p_n] + \dots + \Phi[0, p_1, p_2 + \dots + p_n].
 \end{aligned}$$

From (7) we obtain, for $m=1$

$$\begin{aligned}
 (7') \quad & \Phi[(p_1 + \dots + p_{n-2})q, p_{n-1}q, p_n q] + \dots + \Phi[0, p_1 q, (p_2 + \dots + p_n)q] = \\
 & = \Phi[p_1 + \dots + p_{n-2}, p_{n-1}, p_n] + \dots + \Phi[0, p_1, p_2 + \dots + p_n].
 \end{aligned}$$

(7') implies, for $n=2$

$$(7'') \quad \Phi(0, p_1 q, p_2 q) = \Phi(0, p_1, p_2)$$

for all p_1, p_2, q with $p_1 \geq 0, p_2 \geq 0, 0 < p_1 + p_2 \leq 1, 0 < q \leq 1$; and for $n=3$

$$\begin{aligned}
 (7''') \quad & \Phi(p_1 q, p_2 q, p_3 q) + \Phi(0, p_1 q, (p_2 + p_3)q) = \\
 & = \Phi(p_1, p_2, p_3) + \Phi(0, p_1, p_2 + p_3)
 \end{aligned}$$

for all

$$p_k, q \text{ with } p_k \geq 0, 0 < \sum_{k=1}^3 p_k \leq 1, 0 < q \leq 1 \quad (k=1, 2, 3)$$

As for all (7'') and (7''') are simultaneously true, it follows

$$(8) \quad \Phi(p_1 q, p_2 q, p_3 q) = \Phi(p_1, p_2, p_3)$$

for all

$$p_k, q \text{ with } p_k \geq 0, 0 < \sum_{k=1}^3 p_k \leq 1, 0 < q \leq 1 \quad (k=1, 2, 3)$$

On the other hand, putting

$$p_1 q = p'_1, \quad p_2 q = p'_2, \quad p_3 q = p'_3, \quad \frac{1}{q} = q'$$

from (8) it follows that

$$(8') \quad \Phi(p'_1, p'_2, p'_3) = \Phi(p'_1 q', p'_2 q', p'_3 q')$$

for all

$$p'_k, q' \text{ with } p'_k \geq 0, 0 < \sum_{k=1}^3 p'_k \leq 1, q' \geq 1 \quad (k=1, 2, 3).$$

Now we can notice that the relationships (8) and (8') equal the unique condition

$$(8'') \quad \Phi(p_1 q, p_2 q, p_3 q) = \Phi(p_1, p_2, p_3)$$

for all

$$p_k, q \text{ with } p_k \geq 0, 0 < \sum_{k=1}^3 p_k \leq 1, q > 0 \quad (k=1, 2, 3).$$

Let us suppose now that in (7) $n=2$. (7) itself becomes in such a hypothesis 0

$$(9) \quad \sum_{r=1}^m \Phi[(p_1 + p_2)(w(Q) - q_r), p_1 q_r, p_2 q_r] = \Phi(0, p_1, p_2)$$

We put in (9) $q_r = w(Q)x_r$. Being $q_r \geq 0$ for $1 \leq r \leq m$, $\sum_{r=1}^m q_r = w(Q) > 0$, we will have $x_r \geq 0$, for $1 \leq r \leq m$ and $\sum_{r=1}^m x_r = 1$. Then (9) becomes

$$(9') \quad \sum_{r=1}^m \Phi[(p_1 + p_2)w(Q)(1 - x_r), p_1 w(Q)x_r, p_2 w(Q)x_r] = \Phi(0, p_1, p_2)$$

from which we deduce, bearing in mind (8'')

$$(9'') \quad \sum_{r=1}^m \Phi[(p_1 + p_2)(1 - x_r), p_1 x_r, p_2 x_r] = \Phi(0, p_1, p_2).$$

The relation (9'') having to be true however, if we choose the nonnegative numbers $x_1, \dots, x_m$ such that $\sum_{r=1}^m x_r = q$, and whatever the naturae number m may be, we deduce that necessarily

$$(10) \quad \begin{aligned} & \Phi[(p_1 + p_2)(1 - x_1 - x_2), p_1(x_1 + x_2), p_2(x_1 + x_2)] = \\ & = \Phi[(p_1 + p_2)(1 - x_1), p_1 x_1, p_2 x_1] + \Phi[(p_1 + p_2)(1 - x_2), p_1 x_2, p_2 x_2] \end{aligned}$$

Fixing p_1 and p_2 , $\Phi[(p_1 + p_2)(1 - x), p_1 x, p_2 x]$ is a function of the variable x only, defined over the interval $0 \leq x \leq 1$, there being nonnegative. Putting

$$\varphi(x) = \Phi[(p_1 + p_2)(1 - x), p_1 x, p_2 x]$$

(10) assumes the form

$$(10') \quad \varphi(x_1 + x_2) = \varphi(x_1) + \varphi(x_2)$$

It is known (see [2]), that the only function $\varphi(x)$ satisfying these conditions is of the form

$$\varphi(x) = cx, x \in [0, 1] \text{ with } c \geq 0, \text{ constant.}$$

Correspondingly we obtain

$$(11) \quad \Phi[(p_1 + p_2)(1 - x), p_1 x, p_2 x] = cx$$

From the comparison of (11) with (9'') we deduce, remembering that $\sum_{r=1}^m x_r = 1$

$$c = \Phi(0, p_1, p_2)$$

It is then

$$(11') \quad \Phi[(p_1 + p_2)(1 - x), p_1 x, p_2 x] = \Phi(0, p_1, p_2)x$$

$$p_1 \geq 0, \quad p_2 \geq 0, \quad 0 < p_1 + p_2 \leq 1, \quad 0 \leq x \leq 1.$$

Now we put

$$(p_1 + p_2)(1 - x) = \pi_1, \quad p_1 x = \pi_2, \quad p_2 x = \pi_3$$

from which it results

$$p_1 = (\pi_1 + \pi_2 + \pi_3) \frac{\pi_2}{\pi_2 + \pi_3}, \quad p_2 = (\pi_1 + \pi_2 + \pi_3) \frac{\pi_3}{\pi_2 + \pi_3}, \quad x = \frac{\pi_2 + \pi_3}{\pi_1 + \pi_2 + \pi_3}.$$

We recognize immediately that $0 \leq \pi_r \leq 1$, for $r = 1, 2, 3$ and that $0 < \sum_{r=1}^3 \pi_r \leq 1$.

It is also evident that (11') is equivalent to

$$\Phi(\pi_1, \pi_2, \pi_3) = \frac{\pi_2 + \pi_3}{\pi_1 + \pi_2 + \pi_3} \Phi \left[0, (\pi_1 + \pi_2 + \pi_3) \frac{\pi_2}{\pi_2 + \pi_3}, (\pi_1 + \pi_2 + \pi_3) \frac{\pi_3}{\pi_2 + \pi_3} \right].$$

On the other hand, (8'') and the preceding relationship imply

$$(12) \quad \Phi(\pi_1, \pi_2, \pi_3) = \frac{\pi_2 + \pi_3}{\pi_1 + \pi_2 + \pi_3} \Phi \left(0, \frac{\pi_2}{\pi_2 + \pi_3}, \frac{\pi_3}{\pi_2 + \pi_3} \right)$$

for all

$$\pi_1 : \pi_2 \geq 0, \quad 0 < \sum_{r=1}^3 \pi_r \leq 1 \quad (r = 1, 2, 3).$$

From (3) for $n=3$ we deduce

$$(3') \quad \Phi(p_1, p_2, p_3) + \Phi(0, p_1, p_2 + p_3) = \Phi(p_2, p_1, p_3) + \Phi(0, p_2, p_1 + p_3),$$

$$p_k \geq 0 \quad (k = 1, 2, 3), \quad 0 < \sum_{k=1}^3 p_k \leq 1,$$

and thus, bearing in mind (12)

$$(3'') \quad \begin{aligned} & \frac{p_2 + p_3}{p_1 + p_2 + p_3} \Phi \left(0, \frac{p_2}{p_2 + p_3}, \frac{p_3}{p_2 + p_3} \right) + \Phi \left(0, \frac{p_1}{p_1 + p_2 + p_3}, \frac{p_2 + p_3}{p_1 + p_2 + p_3} \right) = \\ & = \frac{p_1 + p_3}{p_1 + p_2 + p_3} \Phi \left(0, \frac{p_1}{p_1 + p_3}, \frac{p_3}{p_1 + p_3} \right) + \Phi \left(0, \frac{p_2}{p_1 + p_2 + p_3}, \frac{p_1 + p_3}{p_1 + p_2 + p_3} \right) \end{aligned}$$

Let us put

$$p = \frac{p_1}{p_1 + p_2 + p_3}, \quad q = \frac{p_2}{p_1 + p_2 + p_3}, \quad f(p) = \Phi(0, p, 1 - p)$$

The function $f(p)$ is defined over the interval $0 \leq p \leq 1$; it is Lebesgue integrable over this interval, according to F ; by (2) it has the property

$$f(p) = f(1-p) \quad \text{for} \quad p \in [0, 1]$$

It follows further from (3'')

$$(1-p)f\left(\frac{q}{1-p}\right) + f(p) = (1-q)f\left(\frac{p}{1-q}\right) + f(q)$$

for all p, q with $0 \leq p \leq 1, 0 \leq q \leq 1, 0 \leq p+q \leq 1$

This is Tveberg's equation ([3]). It follows that

$$f(p) = -p \log_a p - (1-p) \log_a (1-p) \quad \text{for} \quad p \in [0, 1].$$

From the definition of $f(p)$ and from (12) we then deduce

$$\Phi(p_1, p_2, p_3) = \frac{1}{p_1 + p_2 + p_3} [(p_2 + p_3) \log_a (p_2 + p_3) - p_2 \log_a p_2 - p_3 \log_a p_3]$$

Let us substitute this expression together with (5) in (6), observing that in (5) the constant c is linked to the logarithmic base and that therefore we can assume $I(p) = -\log_b p$. Thus we obtain

$$I(p_1, \dots, p_n) = \frac{-\sum_{k=1}^n p_k \log_a p_k}{\sum_{k=1}^n p_k} + \log_a \left(\sum_{k=1}^n p_k \right) - \log_b \left(\sum_{k=1}^n p_k \right)$$

If¹⁾ $a=b$, then in particular

$$I(p_1, \dots, p_n) = \frac{-\sum_{k=1}^n p_k \log_a p_k}{\sum_{k=1}^n p_k}$$

Thus our statement is proved.

References

- [1] A. RÉNYI, On the foundations of information theory. *Rev. of the Intern. Statistical Institute*, **33** 1 (1965).
- [2] J. ACZÉL, Lectures on functional equations. (1966).
- [3] H. TVERBERG, A new derivation of information function. *Math. Scand.* **6** (1958), 225—29.

(Received February 11, 1968)

¹⁾ It is easy to see, that the condition

$$G) \quad I(p_1, p_2) = I(p)$$

is with the property $a=b$ equivalent.