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On the number of solutions of the generalized Ramanujan-Nagell equation $x^2 - D = k^n$

By XIGENG CHEN (Maoming) and MAOHUA LE (Zhanjiang)

Abstract. Let $D, k \in \mathbb{N}$ be such that $D > 1$, $k > 1$ and $\gcd(2D, k) = 1$. In this paper we prove that the titled equation has at most $3 \cdot 2^{\omega(k)-1} + 1$ positive integer solutions (x, n), where $\omega(k)$ is the number of distinct prime factors of k. Moreover, if max $(D, k) > 10^{60}$, then the equation has at most $3 \cdot 2^{\omega(k)-1}$ solutions (x, n).

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ be the sets of integers, positive integers and rational numbers respectively. Let $D, k \in \mathbb{N}$ be such that $D > 1$, $k > 1$ and $\gcd(D, k) = 1$, and let $\omega(k)$ be the number of distinct prime factors of k. Further, let $N(D, k)$ be the number of solutions (x, n) of the generalized Ramanujan-Nagell equation

$$x^2 - D = k^n, \quad x, n \in \mathbb{N}. \tag{1}$$

There are many works concerned with the upper bounds for $N(D, k)$, including the following:

1 (Beukers [1]). $N(D, 2) \le 4$.

2 (Le [5]). If $D = 2^{2m} - 3 \cdot 2^{m+1} + 1$ for some $m \in \mathbb{N}$ with $m \ge 3$, then $N(D, 2) = 4$. Otherwise, $N(D, 2) \le 3$.

3 (Beukers [2]). If k is an odd prime, then $N(D, k) \le 4$.

4 (Le [4]). If k is an odd prime and $\max(D, k) \ge 10^{240}$, then $N(D, k) \le 3$.

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The last result basically confirmed a conjecture posed by Beukers [2]. In this paper, we extend the result as follows.

Theorem. *If* $\gcd(2D,k) = 1$, *then* $N(D,k) \le 3 \cdot 2^{\omega(k)-1} + 1$. *If moreover* $\max(D,k) > 10^{60}$, *then* $N(D,k) \le 3 \cdot 2^{\omega(k)-1}$.

This last upper bound is best possible while k is an odd prime.

2. Preliminaries

In this section, we assume that $2 \nmid k$, $k \ge 15$ and D is nonsquare.

Lemma 1 ([7, Theorems 1 and 2]). *If the equation*

$$X^2 - DY^2 = k^Z, \quad X, Y, Z \in \mathbb{Z}, \ \gcd(X,Y) = 1, \ Z > 0, \tag{2}$$

is solvable in integers X, Y, Z, *then we have:*

(i) *For a fixed solution* (X,Y,Z), *there exists a unique* $\ell \in \mathbb{N}$ *such that*

$$\begin{gathered} X \equiv \pm \ell Y \pmod{k}, \quad \ell^2 \equiv D \pmod{k}, \\ \ell < \frac{k}{2}, \quad \gcd\left(k, 2\ell, \frac{\ell^2 - D}{k}\right) = 1. \end{gathered} \tag{3}$$

(ii) *All solutions of* (2) *can be put into* $2^{\omega(k)-1}$ *classes in such a way that each solution* (X,Y,Z) *in a class has the same value of* ℓ *in* (3).

(iii) *For a fixed class, say* S, *there exists a unique solution* (X_1, Y_1, Z_1) *in* S *which satisfies* $X_1 > 0$, $Y_1 > 0$, $Z_1 \le Z$ *and*

$$1 < \frac{X_1 + Y_1\sqrt{D}}{X_1 - Y_1\sqrt{D}} < \left(u_1 + v_1\sqrt{D}\right)^2, \tag{4}$$

where Z *runs over all solutions* (X,Y,Z) *in* S, $u_1 + v_1\sqrt{D}$ *is the fundamental solution of the equation*

$$u^2 - Dv^2 = 1, \quad u, v \in \mathbb{Z}. \tag{5}$$

The solution (X_1, Y_1, Z_1) *is called the least solution of* S.

(iv) *If* (X_1, Y_1, Z_1) *is the least solution of* S, *then every solution* (X,Y,Z) *in* S *can be expressed as*

$$\begin{gathered} Z = Z_1 t, \quad X + Y\sqrt{D} = \left(X_1 + \lambda Y_1 \sqrt{D}\right)^t \left(u + v\sqrt{D}\right), \\ t \in \mathbb{N}, \quad \lambda \in \{-1, 1\}, \end{gathered}$$

where (u,v) *is a solution of* (5).

Cleary, if (x,n) is a solution of (1), then $(X,Y,Z)=(x,1,n)$ is a solution of (2).

Lemma 2. *Let (x,n) be a solution of (1). If $(x,1,n)$ belongs to the class S and (X_1,Y_1,Z_1) is the least solution of S, then we have:*

$$n=Z_1t,\quad x+\delta\sqrt{D}=\varepsilon^t\bar{\varrho}^s,\quad \delta\in\{-1,1\},\tag{6}$$

where

$$\begin{aligned}\varepsilon&=X_1+Y_1\sqrt{D},\quad &\bar{\varepsilon}&=X_1-Y_1\sqrt{D},\\ \varrho&=u_1+v_1\sqrt{D},\quad &\bar{\varrho}&=u_1-v_1\sqrt{D},\end{aligned}\tag{7}$$

$s,t\in\mathbb{Z}$ satisfy $t>0,\ t\geq s\geq 0$ and $\gcd(s,t)=1$.

Proof. By (iv) of Lemma 1, (6) holds for some $s,t\in\mathbb{Z}$ with $t>0$. If $s<0$, then

$$\varepsilon^t=X+Y\sqrt{D},\quad \bar{\varrho}^s=u+v\sqrt{D},\quad X,Y,u,v\in\mathbb{N},$$

and $\delta=Xv+Yu\geq 2$, a contradiction. So we have $s\geq 0$. Moreover, by [2, Lemma 3] we have $\gcd(s,t)=1$.

If $\delta=1$, then we have

$$1<\frac{x+\sqrt{D}}{x-\sqrt{D}}=\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^t\bar{\varrho}^{2s}<\varrho^{2t-2s},$$

by (4). Hence $t>s$. If $\delta=-1$, then in view of $\varrho>2\sqrt{D}$ we have

$$\frac{x}{\sqrt{D}}=\left(1+\frac{k^n}{D}\right)^{1/2}>\left\{\begin{matrix}\sqrt{2}, & \text{if } k^n>D\\ 1+k^n/2D>1+1/D, & \text{if } k^n<D\end{matrix}\right\}>\frac{\varrho^2+1}{\varrho^2-1}.$$

Hence

$$\frac{x+\sqrt{D}}{x-\sqrt{D}}=\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{-t}\varrho^{2s}<\varrho^2.$$

Together wih (4) this implies $t+1>s$. So in both cases we obtain $t\geq s$. This completes the proof of Lemma 2. $\square$

Let (x,n), (x',n') be two solutions of (1). If $(x,1,n)$ and $(x',1,n')$ are solutions of (2) which belong to the same class, then this will be denoted by $(x,n)\sim(x',n')$. The pair (D,k) will be called exceptional if

$$k=4a^2+\lambda,\quad D=\left(\frac{k^m-\lambda}{4a}\right)^2-k^m\tag{8}$$

for some $a, m \in \mathbb{N}$, $\lambda \in \{-1, 1\}$ with $m > 1$ and the additional condition $2 \nmid m$ if $\lambda = -1$. If (D, k) satisfies (8), then (1) has three solutions (x, n), (x', n'), (x'', n'') given by

$$(x,n) = \left(\frac{k^m - \lambda}{4a} - 2a, 1\right), \quad (x', n') = \left(\frac{k^m - \lambda}{4a}, m\right),$$
$$(x'', n'') = \left(2ak^m + \lambda\frac{k^m - \lambda}{4a}, 2m+1\right). \tag{9}$$

The solutions in (9) satisfy $(x, n) \sim (x', n') \sim (x'', n'')$.

Lemma 3. *Let (x, n), (x', n'), (x'', n'') be three solutions of* (1) *such that $n < n' < n''$ and $(x, n) \sim (x', n') \sim (x'', n'')$. If (D, k) satisfies* (8), *then we have either* (9) *or $n'' \geq 2n' + \max(3, n, 2n'/3 - 2/3)$. If (D, k) does not satisfy* (8), *then we have $n'' \geq 2n' + \max(3, n, 2n'/3 - 2/3)$.*

Proof. Under the assumptions, $(x, 1, n)$, $(x', 1, n')$ and $(x'', 1, n'')$ are solutions of (2) satisfying

$$x^2 \equiv D \pmod{k^n},\ {x'}^2 \equiv D \pmod{k^{n'}},\ {x''}^2 \equiv D \pmod{k^{n''}},$$

and

$$x \equiv \delta\ell \pmod{k},\ x' \equiv \delta'\ell \pmod{k},\ x'' \equiv \delta''\ell \pmod{k},$$
$$\delta, \delta', \delta'' \in \{-1, 1\},$$

for the same $\ell \in \mathbb{N}$. So we have $x \equiv \pm x' \pmod{k^n}$ and $x' \equiv \pm x'' \pmod{k^{n'}}$. Recalling that $2 \nmid k$ and $k \geq 15$. By much the same argument as in the proof of [2, Lemma 5], we can prove the lemma without any difficulty. □

Let $\alpha = (\log(\varepsilon/\bar{\varepsilon}))/\log \varrho^2$, and let $\Lambda(x, n) = \log\left((x + \sqrt{D})/(x - \sqrt{D})\right)$ for any solution (x, n) of (1). Lemmas 4 and 5 stated below can be proved similarly as Lemmas 9 and 10 of [6], respectively.

Lemma 4. *If (x, n) is a solution of (1) satisfying $k^n \geq 3D$ and* (6), *then s/t is a convergent of α.*

Lemma 5. *Let (x, n), (x', n') be two solutions of* (1) *such that $k^{n'} > k^n \geq 3D$, $(x, n) \sim (x', n')$ and $(x, 1, n)$ belongs to the class S. Further let $(X_1, Y_1, Z_1,)$ be the least solution of S. Then we have $n + n' > Z_1 \log \varrho^2 / \Lambda(x, n)$.*

Lemma 6. *Equation* (1) *has at most one solution* (x,n) *with* $k^n < \sqrt{D}$.

Proof. This follows immediately from [3, Theorem 10.8.2]. $\square$

Lemma 7. *If* (x,n) *is a solution of* (1) *such that* k^n *is a square, then* $k^n < D^2/4$.

Proof. Under the assumption, we have $x + k^{n/2} = D_1$ and $x - k^{n/2} = D_2$, where $D_1, D_2 \in \mathbb{N}$ with $D_1 D_2 = D$. It implies that $k^{n/2} = (D_1 - D_2)/2 \le (D-1)/2 < D/2$. The lemma is proved. $\square$

Lemma 8. *If* (1) *has a solution* (x,n) *such that* k^n *is a non-square and* $k^n \ge 4^{1+s/r} D^{2+s/r}$ *for some* $r, s \in \mathbb{N}$, *then we have*

$$\left|\frac{y}{2k^{n'/2}} - 1\right| > \frac{8}{2187}\left(\frac{81}{4}\right)^{1/s} k^{n/s - n(3+\nu/2) - n'(1+\nu)/2}$$

for any $y, n' \in \mathbb{N}$ *with* $2 \nmid n'$, *where*

$$\nu = \frac{r}{s} + \frac{1}{\log k^n}\left(\log 9 + \frac{r}{s}\log\frac{81}{4}\right).$$

Proof. This follows immediately from [8, Theorem I.2]. $\square$

3. Proof of Theorem

By [2] and [6], it suffices to prove the theorem while k is not a prime power. We may assume that $k \ge 15$. If D is a square, then $D = D_1^2$ and

$$x + D_1 = k_1^n\ , \quad x - D_1 = k_2^n\ , \quad k = k_1 k_2,$$
$$D_1, k_1, k_2 \in \mathbb{N},\ \gcd(k_1, k_2) = 1,$$

by (1). Since the number of such pairs (k_1, k_2) does not exceed $2^{\omega(k)-1}$ we have $N(D,k) \le 2^{\omega(k)-1}$. Hence we may assume also that D is not a square.

Let (x,n) be a solution of (1). Then $(X,Y,Z) = (x,1,n)$ is a solution of (2). By (ii) of Lemma 1, we may assume that $(x,1,n)$ belongs to a certain class S. Let (X_1, Y_1, Z_1) be the least solution of S, and let $N(S)$ be the number of solutions (x,n) of (1) such that $(x,1,n)$ belongs to S. We now suppose that $N(S) > 4$. Then (1) has five solutions (x_i, n_i) $(i = 1, \dots, 5)$ such that $n_1 < \dots < n_5$ and $(x_1, n_1) \sim \dots \sim (x_5, n_5)$. If the pair (D,k) is exceptional and the solutions (x_j,n_j) $(j=1,2,3)$ satisfy (9), then we have

$$\begin{aligned}\Lambda(x_3,n_3)&=\frac{\sqrt{D}}{x_3}=\sum_{i=0}^{\infty}\frac{1}{2i+1}\left(\frac{D}{x_3^2}\right)^i<1.01\frac{\sqrt{D}}{x_3}<\frac{1.01}{8a^2-1}\\&=\frac{1.01}{2k-1}<\frac{1}{28},\end{aligned}\tag{10}$$

by (8). Notice that $k^{n_3}>3D$ by (9). On using Lemma 5 with (10), we get

$$n_3+n_4>28\log\varrho^2>28\log 4D.\tag{11}$$

On the other hand, since $k^{n_3}<(4D)^{1.8}$ by (9), we obtain from (11) that

$$k^{n_4}>(4D)^{55}.\tag{12}$$

By Lemma 7, we see from (12) that k is not a square and $2\nmid n_4n_5$. Let $n=n_4$, $n'=n_5$, $y=2x_5$, $r=1$ and $s=53$. On applying Lemma 8 with (12), we get

$$\left|\frac{x_5}{k^{n_5/2}}-1\right|>\frac{8}{2187}\left(\frac{81}{4}\right)^{1/53}k^{n_4/53-(3+\nu/2)n_4-(1+\nu)n_5/2},\tag{13}$$

where

$$\begin{aligned}\nu&=\frac{1}{53}+\frac{1}{\log k^{n_4}}\left(\log 9+\frac{1}{53}\log\frac{81}{4}\right)\\&<\frac{1}{53}+\frac{1}{55\log 4D}\left(\log 9+\frac{1}{53}\log\frac{81}{4}\right)<0.0364.\end{aligned}\tag{14}$$

Notice that

$$\left|\frac{x_5}{k^{n_5/2}}-1\right|=\frac{D}{k^{n_5/2}\left(x_5+k^{n_5/2}\right)}<\frac{D}{2k^{n_5}}\,.\tag{15}$$

The combination of (13), (14) and (15) yields

$$138Dk^{2.9994n_4}>k^{0.4818n_5}.\tag{16}$$

Since $4D<k^{n_4/55}$ by (12), we get from (16) that

$$6.316n_4>n_5.\tag{17}$$

On the other hand, by Lemma 5 we see from (10) that

$$n_4+n_5>\frac{k^{n_4/2}\log 4D}{1.01\sqrt{D}}>k^{0.496n_4}.\tag{18}$$

The combination of (17) and (18) yields

$$7.316n_4 > k^{0.496n_4}. \tag{19}$$

But (18) is impossible, since $k \geq 15$ and $n_4 > 4$. Therefore, we obtain $N(S) \leq 4$ if the pair (D, k) is exceptional and the solutions (x_j, n_j) $(j = 1, 2, 3)$ satisfy (9).

We now prove the inequality (12) for the other cases. By Lemma 3, if (D, k) satisfies (8) and solutions (x_j, n_j) $(j = 1, 2, 3)$ do not satisfy (9) or (D, k) does not satisfy (8), then $n_3 \geq 2n_2 + 3$. Further, by Lemma 6, we have $k^{n_2} > \sqrt{D}$. Hence $k^{n_3} \geq Dk^3$ and

$$\begin{aligned} \Lambda(x_3, n_3) &= \log \frac{x_3 + \sqrt{D}}{x_3 - \sqrt{D}} = \log \frac{\sqrt{k^{n_3} + D} + \sqrt{D}}{\sqrt{k^{n_3} + D} - \sqrt{D}} \\ &< \log \frac{k^{3/2} + 1}{k^{3/2} - 1} < \frac{1}{28.9855}. \end{aligned} \tag{20}$$

By Lemma 5, we see from (20) that

$$n_3 + n_4 > 28.9855 Z_1 \log \varrho^2 > 28.9855 \log 4D. \tag{21}$$

This follows that

$$k^{n_3 + n_4} > (4D)^{28.9855 \log k}. \tag{22}$$

On the other hand, by Lemma 3, we have $n_4 \geq 2n_3 + 2n_3/3 - 2/3$. Hence $n_3 \leq 3n_4/8 + 1/4$ and

$$k^{n_4 + 2/11} > (4D)^{57.0877}. \tag{23}$$

By (23), if $k^{2/11} \leq (4D)^{2.0877}$, then (12) holds. If $k^{2/11} > (4D)^{2.0877}$, then $k > (4D)^{11.4823}$ and $k^{n_4} > (4D)^{57.4115}$. Therefore, (12) holds in any case. Thus, using the same method, we can prove that

$$N(S) \leq \begin{cases} 4, & \text{if (1) has a solution } (x, n) \text{ such hat} \\ & k^n < \sqrt{D} \text{ and } (x, 1, n) \text{ belongs to } S, \\ 3, & \text{oherwise.} \end{cases} \tag{24}$$

Notice that (1) has at most one solution (x, n) with $k^n < \sqrt{D}$ by Lemma 6. Thus, by (ii) of Lemma 1, we get from (24) that $N(D, k) \leq 3 \cdot 2^{\omega(k)-1} + 1$.

On the other hand, by much the same argument as in the proof of [6, Theorem], we can prove that $N(S) \leq 3$ if $\max(D, k) > 10^{60}$. Then we have $N(D, k) \leq 3 \cdot 2^{\omega(k)-1}$. The proof is complete.

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MAOHUA LE
DEPARTMENT OF MATHEMATICS
ZHANJIANG TEACHERS COLLEGE
P.O. BOX 524048
ZHANJIANG, GUANGDONG
P. R. CHINA

XIGENG CHEN
DEPARTMENT OF MATHEMATICS
MAOMING EDUCATION COLLEGE
P.O. BOX 525000
MAOMING, GUANGDONG
P. R. CHINA

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