

Note on a conjecture of P. Erdős

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The function f defined on the natural numbers is called *additive* if $(m, n) = 1$ implies

$$(1) \quad f(mn) = f(m) + f(n)$$

(as usual, (m, n) denotes the greatest common divisor of m and n). If (1) holds for all pairs m, n then f is called *completely additive*. In [1], P. ERDŐS conjectured that if f is additive and

$$(2) \quad f(n+1) - f(n) = O(1),$$

then there is a constant α such that $f(n) = \alpha \log n + O(1)$.

The following lemma is a partial result which shows that it suffices to prove that if f is completely additive and (2) holds, then $f(n) = \alpha \log n$.

Lemma. *If f is an additive function and $f(n+1) - f(n) = O(1)$, then $g(n) = \lim_{l \rightarrow \infty} \frac{f(n^l)}{l}$ exists. Moreover*

$$f(n) = g(n) + O(1),$$

g is completely additive, and $g(n+1) - g(n) = O(1)$.

PROOF. In the following, the constants in O symbols are absolute; $\zeta(n, k, l)$ and $\theta(n, k, l)$ denote expressions which are bounded, with bounds depending on n and k but not on l .

For arbitrary m and q we have

$$\begin{aligned} f(m^q + m^{q-1} + \dots + 1) &= f(m^q + m^{q-1} + \dots + m) + O(1) = \\ &= f(m^{q-1} + m^{q-2} + \dots + 1) + f(m) + O(1), \end{aligned}$$

whence, by repeated application

$$f(m^q + m^{q-1} + \dots + 1) = q(f(m) + O(1)).$$

Thus, if $(m-1, q) = 1$, then

$$(3) \quad f(m^q - 1) = f(m-1) + f(m^{q-1} + m^{q-2} + \dots + 1) = q(f(m) + O(1))$$

Let n and k be fixed, and let $m = n^k$. If $l > mk$, then we can write l as $l = qk + r$ with $q \equiv 1 \pmod{m-1}$ and $0 \leq r < k(m-1)$. Then by (2) and (3) we have

$$f(n^l) = f(n^l - n^r) + O(n^r) = f(n^{qk} - 1) + \zeta(n, k, l) = q(f(n^k) + O(1)) + \zeta(n, k, l).$$

It follows that

$$\begin{aligned} l^{-1}f(n^l) - k^{-1}f(n^k) &= l^{-1}q(f(n^k) + O(1)) + l^{-1}\zeta(n, k, l) - k^{-1}f(n^k) = \\ &= l^{-1}qO(1) + l^{-1}\zeta(n, k, l) - l^{-1}k^{-1}rf(n^k). \end{aligned}$$

Hence

$$(4) \quad l^{-1}f(n^l) - k^{-1}f(n^k) = (qk + r)^{-1}qO(1) + l^{-1}\theta(n, k, l).$$

Now (4) implies that $\{l^{-1}f(n^l)\}_{l=1}^{\infty}$ is a Cauchy sequence. In fact, first k can be chosen so as to make the first term of the right hand side small, and then it follows that for sufficiently large values of l the corresponding terms of the sequence will have small differences. We therefore may conclude that $g(n) = \lim_{l \rightarrow \infty} l^{-1}f(n^l)$ exists, and by taking limits in (4) we find

$$g(n) = k^{-1}f(n^k) + O(k^{-1}).$$

In particular, for $k=1$ we find

$$f(n) = g(n) + O(1)$$

which, with (2), implies $g(n+1) - g(n) = O(1)$.

Since f is additive, so is g , and it follows from the definition of g that $g(n^k) = kg(n)$ for all n and k , so the function g is completely additive. This completes the proof.

References

- [1] P. ERDŐS, On the distribution function of additive functions, *Ann. of Math.* **47** (1946), 1—20.

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