

New types of topological spaces

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§ 1. Introduction

Our aim, in the present paper, is to introduce some new types of topological spaces (see § 2). The definitions of these spaces are based on those of T_0 , T_1 and T_2 -spaces, (cf. [1]¹).

In § 4 below, we give some examples of these new spaces and in the rest of the paper we discuss some of their interesting properties and relations.

§ 2. The new definitions

Definition 2. 1. A topological space is a T'_0 -space if and only if for each pair of distinct points of the space, there is an open neighborhood of one point to which the other is a boundary point²).

Definition 2. 2. A topological space is a T'_1 -space if and only if for each pair x and y of distinct points of the space, there is an open neighborhood of one point, say x , to which y is a boundary point and a neighborhood of y to which x does not belong. (x is not necessarily a boundary point of the neighborhood of y .)

Definition 2. 3. A topological space is a T''_1 -space if and only if for each pair x and y of distinct points of the space, there is an open neighborhood of x to which y is a boundary point.

*Definition 2. 4.*³) A topological space is a T'_2 -space if and only if for each pair x and y of distinct points of the space, there exists disjoint open neighborhoods of x and y , whose boundary sets are disjoint.

¹) Numbers in square brackets refer to the bibliography at the end.

²) For the definition of the boundary of a subset of a topological space, see for example [1].

³) This definition could equivalently be stated as follows:

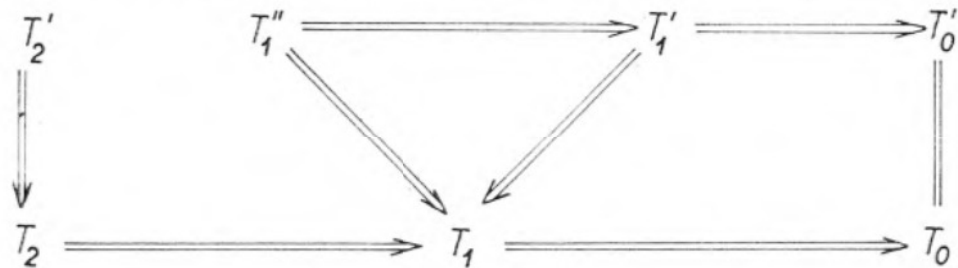
A topological space is a T'_2 -space if and only if, for each pair x and y of distinct points of the space, there exists disjoint closed neighborhoods of x and y .

§ 3. Remarks

From the above definitions one may notice that:

- (i) T'_0 -space is a T_0 -space,
- (ii) T'_1 -space is a $T'_0 + T_1$ -space,
- (iii) T''_1 -space is a T'_1 -space,
- (iv) A topological space with discrete topology satisfies none of the conditions T'_0 , T'_1 and T''_1 , because the boundary of every subset in a discrete topological space is void. Of course, it is a T_2 -space.
- (v) A topological space with indiscrete topology is none of the given new spaces.
- (vi) There is no T'_1 -space or T''_1 -space of finite elements. This because a T''_1 -space or a T'_1 -space is a T_1 -space, but T_1 -space of finite elements is discrete.
- (vii) a T_2 -space is a T_2 -space (a Hausdorff space).

So, one could draw the following diagram:



§ 4. Some examples of the new spaces⁴⁾

We give now the following few examples of T'_0 , T'_1 , T''_1 and T_2 -spaces.

Example 1: The set of all real numbers with the usual topology is a T'_0 , T'_1 , T''_1 , T_2 -space.

Example 2: Let X be a set consisting of four elements a, b, c and d . A topology can be defined in X by taking the open sets to be: (i) X itself, (ii) $\emptyset$ the void set, (iii) the subsets $\{c\}$, $\{b, c\}$, $\{a, b, c\}$ and $\{b, c, d\}$.

The topological space X so defined is a T'_0 -space, but none of the other types.

Example 3: Let X be any infinite set. A topology can be defined in X by taking the open sets to be: (i) X itself, (ii) $\emptyset$; the void set, (iii) any set whose complement is finite. The topological space X is T'_1 but not T_2 .

⁴⁾ The author has not yet found an example of a T'_1 -space which is not a T''_1 -space.

§ 5. Some properties of the new spaces

Theorem 5.1. *A subset of a T'_2 -space i , with the relative topology, is a T'_2 -space.*

PROOF. Let X be a T'_2 -space and A be a subspace of X . Let x and y be any two distinct points of A . Then they are also points of X , and so, since X is T'_2 -space, there are open sets U, V containing x and y respectively such that $\overline{U} \cap \overline{V} = \emptyset$. Let $W = U \cap A$ and $Z = V \cap A$. Then W and Z are open sets in A containing x and y respectively. We have also $\overline{W} \cap \overline{Z} = \emptyset$. Let⁵⁾ $\tilde{W} = \overline{W} \cap A$ and $\tilde{Z} = \overline{Z} \cap A$. Then $\tilde{W} \cap \tilde{Z} = (\overline{W} \cap A) \cap (\overline{Z} \cap A) = (\overline{W} \cap \overline{Z}) \cap A = \emptyset \cap A = \emptyset$. Therefore A is a T'_2 -space.

Theorem 5.2. *An open subset of a T''_1 -space (T'_0, T'_1 -space) is, with the relative topology, a $T''_1(T'_0, T'_1)$ -space.*

The proof of this theorem becomes direct after having the following lemma.

Lemma 5.3. *Let (X, τ) be a topological space, (Y, τ') be an open subspace, and let $A \subset X$ be an open subset. If $y \in Y$ is a τ -boundary point of A , then y is a τ' -boundary point of $A \cap Y$.*

PROOF. If $y \in Y$ is a τ -boundary point of A , then every τ -neighborhood of y has a non-empty intersection with A . Since $Y \subset X$ is open in X , every τ' -neighborhood of y is a τ -neighborhood. Hence, every τ -neighborhood of y intersects $A \cap Y$. This implies that y is a τ' -boundary point of $A \cap Y$.

PROOF OF THEOREM 5.2. Let (X, τ) be a T''_1 -space and let (Y, τ') be any open subspace. For any two distinct points $y_1, y_2 \in Y$ there exists two τ -open neighborhoods V_1 and V_2 of y_1 and y_2 respectively, such that y_1 is a τ -boundary point of V_2 and y_2 is a τ -boundary point of V_1 . The intersections of V_1 and V_2 with Y : $W_1 = V_1 \cap Y$ and $W_2 = V_2 \cap Y$, are two τ' -open neighborhoods of y_1 and y_2 respectively. By the above lemma y_1 is a τ' -boundary point of W_2 and y_2 is a τ' -boundary point of W_1 . Hence (Y, τ') is a T''_1 -space.

Theorem 5.4. *A T'_0 -space X which is also a T_1 -space is a T'_1 -space.*

PROOF. Let $x, y \in X$ be any two distinct points. Suppose that one of these points, say x , has an open neighborhood U and y is a boundary point of it. Since X is a T_1 -space, one may consider x as a closed subset of X , and so $V = X - \{x\}$ is an open neighborhood of y which does not contain x . Hence X is a T'_1 -space.

Theorem 5.5. *A connected T_1 -space is T''_1 -space.*

PROOF. Let X be a connected T_1 -space and let $x, y \in X$ be any two distinct points. Then $X - \{y\}$ and $X - \{x\}$ may be considered as two open neighborhoods of x and y respectively. Since X is connected,

$$\overline{X - \{x\}} = \overline{X - \{y\}} = X.$$

So x is a boundary point of $X - \{x\}$ and y is a boundary point of $X - \{y\}$. Hence X is a T''_1 -space.

⁵⁾ $\sim$ is the closure operation with respect to the relative topology.

We give now an example to show that the converse of this theorem is not necessarily true.

Example: The rationals, with the usual topology of the reals relativized is a T_1'' -space but it is not a connected space.

Remark. Of course, if X is infinite, and τ is the smallest topology such that (X, τ) is a T_1 -space, then (X, τ) is connected and therefore (X, τ) is a T_1'' -space.

Theorem 5. 6. *A T_4 -space is⁶⁾ T_2' .*

PROOF. Let X be a T_4 -space and let $x, y \in X$ be any two distinct points.

Since X is a T_1 -space, we may consider x and y as two disjoint closed sets in X . As X is assumed to be normal, there are two disjoint open sets U, V such that $x \in U$ and $y \in V$. Hence, we get two disjoint open sets U_1 and V_1 (see [2]) such that $x \in U_1$, $y \in V_1$ and $\bar{U} \subset U$, $\bar{V}_1 \subset V$. So $\bar{U} \cap \bar{V}_1 = \emptyset$ and consequently X is T_2' -space.

Note: More interesting properties of these new spaces will be discussed in a coming paper.

References

- [1] J. L. KELLEY, General Topology, New York (1955).
- [2] E. M. PATTERSON, Topology, London (1956).

(Received Mai 25, 1968.)

⁶⁾ A T_4 -space is one which is normal and T_1 (cf. [1]).