

On J_n -groups

By F. FENYVES (Debrecen)

1. Introduction. We start with the following definition. Let G be a group, and let n be a positive integer. An endomorphism α of G is said to be a J_n -endomorphism if α satisfies the J_n -identity

$$\begin{aligned} & ((g_1 g_2 \cdots g_n)^{\alpha} g_{n+1})^{\alpha} ((g_2 g_3 \cdots g_n g_{n+1})^{\alpha} g_1)^{\alpha} \cdots ((g_i g_{i+1} \cdots g_n g_{n+1} g_1 \cdots g_{i-2})^{\alpha} g_{i-1})^{\alpha} \cdots \\ & \cdots ((g_n g_{n+1} g_1 \cdots g_{n-2})^{\alpha} g_{n-1})^{\alpha} ((g_{n+1} g_1 \cdots g_{n-1})^{\alpha} g_n)^{\alpha} = e \end{aligned}$$

for all $g_1, g_2, \dots, g_{n+1}$ in G , where g_i^{α} denotes the image of g_i under α and e is the identity of G . (A J_n -automorphism is defined in the obvious way.) It is easy to see that the trivial endomorphism θ defined by $g^{\theta} = e$ for all g in G is a J_n -endomorphism. Also, it is easy to check that the mapping $\alpha: g \rightarrow g^{-1}$ is a J_1 -endomorphism of G if (and only if) G is an abelian group.

A group G which admits a nontrivial J_n -endomorphism is said to be a J_n -group. For example, every abelian group is a J_1 -group.

The J_2 -groups were first considered by B. M. PUTTASWAMAIAH in [2]. In [1] J. MORGADO proved that the group G is a J_2 -group if and only if G is a semidirect product of a proper normal subgroup and an abelian group with the unique square root property.

The purpose of this article is to extend the above theorem to J_n -groups. We also extend the other results of [1] and [2] to the general case.

2. Preliminary lemmas. Recall that a group G has the unique n th root property, if for every g in G the equation $x^n = g$ has a unique solution in G .

Lemma 1. *The following properties of a torsion group are equivalent:*

- (i) G has the unique n th root property;
- (ii) $(\text{ord } g, n) = 1$ for every g in G .

PROOF. (i) implies (ii). Suppose, to obtain a contradiction, that there exists an element g in G such that $(\text{ord } g, n) > 1$. Then we can choose a prime p for which $\text{ord } g = pk$ and $n = pl$ where k, l are positive integers. Hence we must have $g^k \neq e$ as an element of order p . Since $p \nmid n$, it results in $e = (g^k)^p = (g^k)^n = e^n$, which yields, by (i), $g^k = e$. This contradiction completes the proof of (i) implies (ii).

(ii) implies (i). Consider g in G , since $(\text{ord } g, n) = 1$ there exist integers k and l such that $kn = l \cdot \text{ord } g + 1$. If there exists an element x in G such that $x^n = g$, then $x^{n \cdot \text{ord } g} = (x^n)^{\text{ord } g} = e$ which implies, by $(\text{ord } x, n) = 1$, $\text{ord } x \mid \text{ord } g$ i.e. $x^{\text{ord } g} = e$.

Therefore $x = x^{l \cdot \text{ord } g} x = x^{nk} = g^k$, that is we must have $x = g^k$. Conversely, $x = g^k$ is a solution of the equation $x^n = g$. In fact, one obtains $(g^k)^n = g^{l \cdot \text{ord } g + 1} = g$.

Notation. If α and β are mappings of G into G , then $g^{\alpha+\beta}$ denotes $g^\alpha g^\beta$ and $g^{\alpha\beta}$ denotes $(g^\alpha)^\beta$.

Lemma 2. *Let α be an endomorphism of the group G . Then the following conditions are necessary and sufficient in order that α be a J_n -endomorphism of G :*

$$(1) \quad g_1^{n\alpha^2 + \alpha} = e \quad \text{for all } g_1 \text{ in } G;$$

$$(2) \quad g_1^{\alpha^2} g_2^{\alpha^2} = g_2^{\alpha^2} g_1^{\alpha^2} \quad \text{for all } g_1, g_2 \text{ in } G.$$

PROOF. Let us begin by supposing that α is a J_n -endomorphism of G . The J_n -identity can be rewritten in the following equivalent form

$$(*) \quad (g_1^{\alpha^2} g_2^{\alpha^2} \dots g_n^{\alpha^2} g_{n+1}^\alpha) (g_2^{\alpha^2} g_3^{\alpha^2} \dots g_{n+1}^{\alpha^2} g_1^\alpha) \dots \\ \dots (g_i^{\alpha^2} g_{i+1}^{\alpha^2} \dots g_n^{\alpha^2} g_{n+1}^{\alpha^2} g_1^{\alpha^2} \dots g_{i-2}^{\alpha^2} g_{i-1}^\alpha) \dots (g_{n+1}^{\alpha^2} g_1^{\alpha^2} \dots g_{n-1}^{\alpha^2} g_n^\alpha) = e$$

for all $g_1, g_2, \dots, g_{n+1}$ in G .

Then taking $g_2 = g_3 = \dots = g_{n+1} = e$ in $(*)$ we see that $g_1^{\alpha^2} g_1^\alpha g_1^{(n-1)\alpha^2} = e$ whence follows

$$(1) \quad g_1^{n\alpha^2 + \alpha} = e \quad \text{for all } g_1 \text{ in } G.$$

This of course implies

$$(3) \quad g_1^{\alpha^2} g_1^\alpha = g_1^\alpha g_1^{\alpha^2} \quad \text{for all } g_1 \text{ in } G.$$

Also, it follows from $(*)$, by $g_3 = g_4 = \dots = g_{n+1} = e$,

$$g_1^{\alpha^2} g_2^{\alpha^2} g_2^\alpha g_1^\alpha g_2^{\alpha^2} g_2^\alpha (g_1^{\alpha^2} g_2^{\alpha^2})^{n-2} = e.$$

But this can be written as

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} (g_1 g_2)^\alpha (g_1 g_2)^{(n-2)\alpha^2} = e$$

in view of (3). Thus

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} = (g_1 g_2)^{-(n-2)\alpha^2 - \alpha}.$$

Using (1), we express this relation as

$$(g_1 g_2)^{\alpha^2} (g_2 g_1)^{\alpha^2} = (g_1 g_2)^{2\alpha^2}$$

and this implies that $(g_2 g_1)^{\alpha^2} = (g_1 g_2)^{\alpha^2}$. Whence one gets

$$(2) \quad g_1^{\alpha^2} g_2^{\alpha^2} = g_2^{\alpha^2} g_1^{\alpha^2} \quad \text{for all } g_1, g_2 \text{ in } G.$$

Conversely, suppose that (1) and (2) hold. First of all, we show that

$$(4) \quad g_1^\alpha g_2^{\alpha^2} = g_2^{\alpha^2} g_1^\alpha \quad \text{for all } g_1, g_2 \text{ in } G.$$

In fact, it follows from (1) that $g_1^{n\alpha^2 + \alpha} g_2^{n\alpha^2 + \alpha} = e$. Using (3) and (2) we express this relation as

$$g_1^\alpha g_2^{\alpha^2} g_1^{n\alpha^2} g_2^{(n-1)\alpha^2 + \alpha} = e,$$

which implies, since $g_1^{nz^2} = g_1^{-z}$ and $g_2^{(n-1)z^2+z} = g_2^{-z^2}$,

$$g_1^z g_2^{z^2} g_1^{-z} g_2^{-z^2} = e$$

that is (4) holds.

Then by repeated application of (4), (2) and (1) we have

$$\begin{aligned} & (g_1^{z^2} g_2^{z^2} \cdots g_n^{z^2} g_{n+1}^z)(g_2^{z^2} g_3^{z^2} \cdots g_{n+1}^{z^2} g_1^z) \cdots (g_{n+1}^{z^2} g_1^{z^2} \cdots g_{n-1}^{z^2} g_n^z) = \\ & = (g_1^{z^2} g_1^z g_1^{(n-1)z^2})(g_2^{z^2} g_2^z g_2^{(n-2)z^2}) \cdots (g_k^{z^2} g_k^z g_k^{(n-k)z^2}) \cdots \\ & (g_n^{nz^2} g_n^z)(g_{n+1}^z g_{n+1}^{nz^2}) = \prod_{i=1}^{n+1} g_i^{nz^2+z} = e \text{ for all } g_1, g_2, \dots, g_{n+1} \text{ in } G. \end{aligned}$$

Consequently we have proved that α is a J_n -endomorphism of G .

3. Results.

Theorem 1. *If α is a J_n -endomorphism of a group G then*

- (i) G^z is an abelian group,
- (ii) the restriction of α to G^z is an automorphism of G^z ,
- (iii) G^z has the unique n th root property.

PROOF. (i) In fact, if $a, b \in G^z$ then there exist g, h in G such that $g^z = a$ and $h^z = b$. Thus using (1) and (2) we get

$$ab = g^z h^z = g^z (h^{-n})^{z^2} = (h^{-n})^{z^2} g^z = h^z g^z = ba.$$

(ii) First, it is clear that $G^{z^2} \subseteq G^z$, because $g^{z^2} = (g^z)^z$ for all g in G . Secondly, if a is an arbitrary element of G^z then exists $g \in G$ such that $g^z = a$. Then it follows from (1) that $a = g^z = (g^{-n})^{z^2}$, and this implies $G^z \subseteq G^{z^2}$. Hence the restriction of α to G^z is an epimorphism. Next, let $a (= g^z), b (= h^z) \in G$. Then it follows from $a^z = b^z$ that $g^{z^2} = h^{z^2}$ i.e. $(g^{-1}h)^{z^2} = e$. This combined with (1) implies $e = ((g^{-1}h)^{z^2})^{-n} = (g^{-1}h)^{-nz^2} = (g^{-1}h)^z = (g^{-1})^z h^z$, and so $g^z = h^z$ that is $a = b$. In other words, the restriction of α to G^z is also a monomorphism, and thus it is an automorphism.

(iii) Consider $a = g^z$ in G^z . Then it follows from (1) that $a = g^z = g^{-nz^2} = ((g^{-z})^z)^n$. This means that $x = g^{-z^2}$ is a solution of equation $x^n = a$ in G^z . This solution is unique. Indeed, if x_1 and x_2 are solutions of the equation $x^n = a$ in G^z then $x_1^{-1}x_2$ is a solution of the equation $x^n = e$ in G^z . But the equation $x^n = e$ has only the solution $x = e$ in G^z . In fact, if $(y^z)^n = e$ for some $y \in G$, then using (1) we obtain $e = ((y^z)^n)^z = y^{nz^2} = y^{-z}$. Hence $y^z = e$ and so $x_1 = x_2$. With this the validity of (iii) is proved.

Theorem 2. *The following three assertions concerning a group G are equivalent.*

- (A) G has exactly one J_n -automorphism.
- (B) G has a J_n -automorphism.
- (C) G is an abelian group and has the unique n th root property.

PROOF. That (A) implies (B) is trivial; that (B) implies (C) has been proved already by Theorem 1. The proof will be complete when we show that (C) implies (A).

Next we suppose that (C) holds. Let $\alpha: G \rightarrow G$ be the mapping defined by $g^\alpha = (g^{-1})^{\frac{1}{n}}$ for all g in G , where $g^{\frac{1}{n}}$ denote the unique n th root of g . We shall prove that α is a J_n -automorphism of G . First, α is onto; because if $g \in G$, then $(g^{-n})^\alpha = (g^n)^{\frac{1}{n}} = g$. Secondly, α is one to one; for if $g_1^\alpha = g_2^\alpha$, then $g_1^{-1} = ((g_1^{-1})^{\frac{1}{n}})^n = ((g_2^{-1})^{\frac{1}{n}})^n = g_2^{-1}$, and so $g_1 = g_2$. Next, α is an automorphism of G . Let $g_1, g_2 \in G$. Suppose $(g_1^{-1})^{\frac{1}{n}} = h_1, (g_2^{-1})^{\frac{1}{n}} = h_2$ i.e. $g_1^{-1} = h_1^n, g_2^{-1} = h_2^n$. Then $(g_1 g_2)^{-1} = g_1^{-1} g_2^{-1} = h_1^n h_2^n = (h_1 h_2)^n$, hence

$$(g_1 g_2)^\alpha = ((g_1 g_2)^{-1})^{\frac{1}{n}} = h_1 h_2 = (g_1^{-1})^{\frac{1}{n}} (g_2^{-1})^{\frac{1}{n}} = g_1^\alpha g_2^\alpha.$$

Lastly, it results $g^{n\alpha^2 + \alpha} = ((g^\alpha)^\alpha)^n g^\alpha = (((g^\alpha)^{-1})^{\frac{1}{n}})^n g^\alpha = (g^\alpha)^{-1} g^\alpha = e$ for all g in G , and so, by Lemma 2, α is a J_n -automorphism of G . On the other hand, if β were a second J_n -automorphism of G , then we prove that $\alpha = \beta$. Indeed, one has clearly

$$e = (((g^{-1})^\beta g^\beta)^\beta)^n = g^{-n\beta^2} (g^\beta)^{n\beta} = g^\beta (g^\beta)^{n\beta} = (g(g^\beta)^n)^\beta.$$

So we find that $g(g^\beta)^n = e$, hence $g^\beta = (g^{-1})^{\frac{1}{n}} = g^\alpha$ for all g in G , which means that $\alpha = \beta$. Consequently (C) implies (A).

Using Lemma 1, we now establish the following.

Corollary 1. *If G is a torsion group, then there exists a J_n -automorphism of G if and only if G is abelian and $(\text{ord } g, n) = 1$ for all g in G .*

In particular, if G is an abelian group of exponent k then there exists a J_n -automorphism of G if and only if $(k, n) = 1$.

Corollary 2. *The following are equivalent conditions on a group G .*

- (A) G is a J_n -group under an inner automorphism of G .
- (B) G is a J_n -group under the identity mapping of G .
- (C) G is an abelian group of exponent $n+1$.

We now can formulate a characterization of J_n -groups.

Theorem 3. *For any group G the following two statements are equivalent.*

- (A) G is a J_n -group.
- (B) G is the semi-direct product of a proper normal subgroup B by an abelian subgroup A having the unique n th root property.

PROOF. Assuming (A), let α be a non-trivial J_n -endomorphism of G . Then it holds that the quotient group $G/\text{Ker } \alpha$ is isomorphic to G^α . Moreover for all g in G we have $(g^{n\alpha} g)^\alpha = g^{n\alpha^2 + \alpha} = e$. Hence $g^{n\alpha} g \in \text{Ker } \alpha$, that is $g \text{ Ker } \alpha = g^{-n\alpha} \text{Ker } \alpha$ and here $g^{-n\alpha} \in G^\alpha$. On the other hand, if h^α were a second element of G^α for which $g \text{ Ker } \alpha = h^\alpha \text{Ker } \alpha$, then $h^{-\alpha} g \in \text{Ker } \alpha$. Therefore $h^{-\alpha^2} g^\alpha = e$, hence $e = h^{-n\alpha^2} g^{n\alpha} = h^\alpha g^{n\alpha}$, which implies $h^\alpha = g^{-n\alpha}$. Consequently (A) $\Rightarrow$ (B).

Conversely, assume (B). Define a mapping α of G into G by $\alpha = \beta\gamma\delta$, where

- (i) β is the natural homomorphism of G onto G/B ;

(ii) γ is the isomorphism of G/B onto A defined by $(gB)^\gamma = g^*$, where g^* is the unique element of A for which $gB = g^*B$; and

(iii) δ is the unique J_n -automorphism of A .

Then the next facts are obvious: α is a nontrivial endomorphism of G ; $G^\alpha = A$; and $\text{Ker } \alpha = B$.

Moreover, one has

$$g^{n\alpha} = (g^\alpha)^n = (g^{\beta\gamma\delta})^n = ((gB)^\gamma)^\delta)^n = (g^*)^n = (g^*)^{-1}.$$

This implies, by (ii), that $gB = g^{-n\alpha}B$. Hence it follows that $g^{n\alpha}g \in B$. Therefore we have

$$(1) \quad e = (g^{n\alpha}g)^\alpha = g^{n\alpha^2 + \alpha} \quad \text{for all } g \text{ in } G.$$

Lastly, recalling that $\text{Ker } \alpha$ is an abelian group, we also have

$$(2) \quad g^{x^2}h^{x^2} = h^{x^2}g^{x^2} \quad \text{for all } g, h \text{ in } G.$$

Thus we have verified (see Lemma 2) that α is a nontrivial J_n -endomorphism of G , proving that (B) implies (A).

References

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