Abelian pseudo lattice ordered groups

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I. Introduction

Throughout this paper only (additive) abelian groups will be considered. An *o-ideal* C of a *po*-group G is a directed subgroup of G such that $0 \leqq g \leqq c \in C$ and $g \in G$, imply $g \in C$. A *value* of $0 \neq g \in G$ is an o-ideal M of G which is maximal with respect to $g \notin M$. Let

$$M(g) = \{M \subseteqq G \mid M \text{ is a value of } g\} \text{ and } M^*(g) = \bigcap M(g).$$

Two positive elements $a, b \in G$ are *pseudo disjoint* (p-disjoint) if $a \in M^*(b)$ and $b \in M^*(a)$, and G is a *pseudo lattice ordered group* (p-group) if each $g \in G$ has a representation $g = a - b$, where a and b are p-disjoint.

Throughout this paper G will always denote an abelian p-group

The concept of a p-group was introduced in [2] and we shall make use of the theory developed there. In particular, $a, b \in G^+ = \{g \in G \mid g \geqq 0\}$ are p-disjoint if and only if

(*) $\qquad c \leqq a$ and b implies $nc \leqq a$ and b for all $n > 0$.

Thus, each lattice ordered group (l-group), and hence each totally ordered group (o-group) is a p-group.

In [5], the main result asserts that G is also a Riesz group. Our first result shows that (*) is sufficient for a Riesz group H to be a p-group. Moreover, it is shown in [5] that, if a and b are p-disjoint in G, then $\{0 \leqq m \in G \mid m \leqq a \text{ and } b\}$ is a convex subsemigroup of G^+ and hence, is the positive cone for an o-ideal

$$H(a, b) = [\{0 \leqq m \in G \mid m \leqq a \text{ and } b\}]$$

where $[S]$ denotes the subgroup generated by the subset S of G. Also, $H(a, b) \subseteqq M^*(a) \cap M^*(b)$, and clearly, G is an l-group if and only if $H(a, b) = 0$ for each pair of p-disjoint elements a, b of G. Most of the results in this paper point up the similarity between p-groups and l-groups. The measure of the difference is the set of o-ideals $H(a, b)$.

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Let τ be a homomorphism of a *po*-group A into a *po*-group B. We say τ is an *o-homomorphism* if

$$(A\tau)\cap B^+ \subseteq (A^+)\,\tau \subseteq B^+.$$

If A and B are *p*-groups, then τ is a *p-homomorphism* if τ maps *p*-disjoint pairs onto *p*-disjoint pairs. Each *p*-homomorphism τ is an *o*-homomorphism and so, if τ is one-to-one, then both τ and τ^{-1} preserve order. In section 2 we derive the standard isomorphism theorems for *p*-groups.

In section 3 we introduce the concept of the positive and negative parts of an element of G and also the absolute value of such an element, namely,

$$g^+ = a+H(a, b), \qquad g^- = b+H(a, b), \qquad |g| = g^+ + g^- = a+b+H(a, b)$$

where $g = a-b$ with a and b *p*-disjoint. These definitions are independent of a and b, and if G is an *l*-group then these are the usual definitions. Also, most of the usual properties of these concepts for *l*-groups remain true for *p*-groups.

In section 4 we investigate the *o*-ideal H of G that is generated by all the *o*-ideals $H(a, b)$. For example, if K is an *o*-ideal of G, then G/K is an *l*-group if and only if $K \supseteq H$. We also show, that each *p*-group G is a *p*-subgroup of a *p*-group V, where V^+ is the union of lattice cones, and then investigate when G^+ is the union of lattice cones.

In section 5 we show that a reasonably large class of *p*-groups are *o*-homomorphic images of *l*-groups.

Theorem 1. 1. *For a Riesz group H, the following are equivalent.*

(i) *H is a p-group.*

(ii) *Each $h\in H$ has a representation $h = a-b$, where a and $b\in H^+$ and $c\leqq a$ and b implies $nc\leqq a$ and b for all $n>0$.*

(iii) *For each $g\in H$, there is $a\in H^+$ such that $g\leqq a$, and whenever 0 and $g\leqq x$, then $a \leqq x+h$ for some $h\in M^*(a)\cap M^*(a-g)$.*

PROOF. Let H be a Riesz group. By Theorem 4. 5 of [2] we have (i) implies (ii) and by Theorem 3. 1 of [5], (iii) implies (i). To complete the proof, suppose $g\in H$ and $g = a-b$ where a and b satisfy the conditions of (ii). Then $a\in H^+$ and $g\leqq a$. If 0 and $g \leqq x\in H$, then H, a Riesz group, implies there is $z\in H$ such that 0 and $g\leqq z\leqq a$ and x. Let $h = a-z\geqq 0$, then $x+h \geqq a$.

If $h=0$, then $a\leqq x$ so suppose $h>0$. Clearly, $a\geqq h$ and since $z\geqq g$, $h = a-z \leqq$ $\leqq a-g = b$ so $h\leqq a$ and b. By (ii), $nh\leqq a$ and b for all $n>0$. If M is a value of a and $h\notin M$, then $M < M+h$ and $M+nh = n(M+h) \leqq M+a$ for all $n>0$. But by (3. 4) of [2], M'/M is *o*-simple, where M' is the intersection of all *o*-ideals of H that properly contain M. Since $M+a \in M'/M$, it follows that $M+a \leqq n(M+h)$ for some $n>0$, a contradiction. Thus, $h\in M$. A similar argument for b yields $h\in M^*(a)\cap M^*(b) = M^*(a)\cap M^*(a-g)$.

II. The isomorphism theorems for p-groups

We denote by $O(G)$, the set of all o-ideals of G.

Theorem 2. 1. *The set $O(G)$ is a complete distributive sublattice of the lattice of all subgroups of G. Moreover,*

$$A\wedge(\vee_\Gamma B_\gamma) = \vee_\Gamma(A\wedge B_\gamma) \quad \text{for} \quad A, B_\gamma \in O(G).$$

PROOF. By Theorem 4. 3 in [2], $O(G)$ is closed with respect to arbitrary intersection. This theorem now follows from Theorem 5. 6 in [4] which asserts that for a Riesz group, $O(G)$ is a distributive sublattice of the lattice of all subgroups of G.

Proposition 2. 2. *Suppose $K\in O(G)$.*

(i) *If a and b are p-disjoint in G, then $K+a$ and $K+b$ are p-disjoint in G/K and $H(K+a, K+b) = \dfrac{K+H(a, b)}{K}$.*

(ii) *If X and Y are p-disjoint in G/K, then $X = K+u$ and $Y = K+v$ where u and v are p-disjoint in G.*

PROOF. (i) If a and b are p-disjoint in G, then $K+a$ and $K+b$ are p-disjoint in G/K by (ii) of Theorem 1. 1 and

$$\big(K+H(a, b)\big)/K = \{K+x \,|\, x\in H(a, b)\} = [\{K+x\,|\,0 \leqq x\in H(a, b)\}].$$

If $0 \leqq x \in H(a, b)$, then $x\leqq a$ and b so $K\leqq K+x \leqq K+a$ and $K+b$. Therefore, $K+x\in H(K+a, K+b)$. Conversely, if $K < X\in H(K+a, K+b)$ where $X = K+x$ and $0<x \in G$, then $K < K+x \leqq K+a$ and $K+b$, so there exists $k_1, k_2\in K$ such that $k_1+x \leqq a$ and $k_2+x \leqq b$. Since K is directed, there is $k\in K$ such that $k\leqq k_1$ and k_2 and hence, $k+x$ and $0\leqq a$ and b. Also, there is $z\in G$ such that 0 and $k+x \leqq z\leqq a$ and b. It follows that $z\in H(a, b)$ and $K<K+x \leqq$ $\leqq K+z\in (K+H(a, b))/K$ which is convex, so $X = K+x\in \big(K+H(a, b)\big)/K$. (ii) Let $X = K+x$ and $Y = K+y$ be p-disjoint in G/K with $0\leqq x$ and y in G, and $x-y = a-b$ where a and b are p-disjoint in G. Then $K+a$ and $K+b$ are p-disjoint, $K+x = K+a+K+m$ and $K+y = K+b+K+m$ where $K+m \in H(K+a, K+b) = \big(K+H(a, b)\big)/K$. (See [5].) So, without loss of generality, $m\in H(a, b)$ and hence, $u = a+m$ and $v = b+m$ are p-disjoint.

Remark. One should now be able to prove that if $X_1, \ldots, X_n$ are (pairwise) p-disjoint in G/K, then there are p-disjoint elements $x_1, \ldots, x_n$ in G such that $X_i = K+x_i$ for $1\leqq i\leqq n$, but we have not been able to do so.

Induced Homomorphism Theorem. *Let A, B, C and D be p-group and α, β and δ be p-homomorphisms such that*

$$\begin{tikzcd}
D \arrow[r, dashed, "\alpha^*"] & C \\
A \arrow[u, "\delta"] \arrow[r, "\alpha"] & B \arrow[u, "\beta"']
\end{tikzcd}$$

Further suppose that δ is onto and that $K(\delta)\alpha\subseteq K(\beta)$, where $K(\delta) = kernel\ \delta$.

(a) *There exists a unique p-homomorphism α^* of D into C so that the diagram commutes.*

(b) *α^* is an o-isomorphism if and only if $K(\delta) \subseteq K(\beta)\alpha^{-1}$.*

PROOF. This is a standard result from group theory so we need only show α^* is a p-homomorphism. If x and y are p-disjoint in D, then by the last proposition, there exist p-disjoint elements a and b in A such that $a\delta = x$ and $b\delta = y$. Thus, $x\alpha^* = a\delta\alpha^* = a\alpha\beta$ and $y\alpha^* = b\delta\alpha^* = b\alpha\beta$ are p-disjoint in C.

Corollary 1. *If $A, B \in O(G)$ and $A \subseteq B$, then B/A is an o-oideal of G/A and the natural isomorphism of G/B onto $(G/A)/(B/A)$ is a p-isomorphism.*

PROOF. Clearly, B/A is an o-ideal of G/A and the natural homomorphisms α, β and δ are p-homomorphisms.

$$\begin{tikzcd} G/B & (G/A)/(B/A) \\ G \arrow[u, "\delta"] \arrow[r, "\alpha"] & G/A \arrow[u, "\beta"'] \end{tikzcd}$$

Moreover, $K(\delta)\alpha = B\alpha = B/A = K(\beta)$ and $K(\beta)\alpha^{-1} = (B/A)\alpha^{-1} = B = K(\delta)$. The corollary follows by (b) of the theorem.

Corollary 2. *If $A, B \in O(G)$, then the natural isomorphism of $(A+B)/A$ onto $B/(A \cap B)$ is a p-isomorphism.*

PROOF. $A+B \in O(G)$ and hence, is a p-group. It follows that $(A+B)/A$ and $B/(A \cap B)$ are p-groups and we have

$$\begin{tikzcd} B/(A\cap B) & (A+B)/A \\ B \arrow[u, "\delta"] \arrow[r, "\alpha"] & A+B \arrow[u, "\beta"'] \end{tikzcd}$$

where, $K(\delta)\alpha = (A \cap B)\alpha = A \cap B \subseteq A = K(\beta)$ and $K(\beta)\alpha^{-1} = A \cap B = K(\delta)$.

Definition. A subgroup K of G is a *p-subgroup* if each $k \in K$ has a representation $k = a - b$, where a and b are p-disjoint in G and belong to K.

In [2], Theorem 4. 3, it is shown that each $M \in O(G)$ is a p-subgroup. In fact, if $g = a - b \in M$ where a and b are p-disjoint in G, then a and $b \in M$ and are p-disjoint in M. If π is a p-homomorphism of a p-group A into a p-group B, then clearly, $A\pi$ is a p-subgroup of B.

Lemma 2. 3. (i) *If M is a p-subgroup of G (or merely directed), then $M\sigma = \{x \in G \mid a \leqq x \leqq b$, for $a, b \in M\}$ is the o-ideal of G generated by M.* (ii) *If $A \in O(G)$ and B is a p-subgroup of G, then $A \cap B \in O(B)$.*

PROOF. (i) Clearly, $M\sigma$ is a convex subgroup of G that contains M. If $x \in M\sigma$, then $x \leqq b \in M$ and since M is directed, there is $m \in M$ such that 0 and $b \leqq m$. Thus,

$M\sigma$ is directed and $M\sigma$ is an o-ideal of G. Clearly, each o-ideal containing M must contain $M\sigma$.

(ii) If $x \in A \cap B$, then $x = u - v$ where u and v are p-disjoint in G and belong to B. But $A \in O(G)$ so u and $v \in A$ and $A \cap B$ is directed. Moreover, if $0 < x < y \in A \cap B$ and $x \in B$, then $x \in A \cap B$ and so $A \cap B$ is convex in B.

Remark. If $A \in O(G)$ and B is a p-subgroup, then is $A + B$ a p-subgroup? If so, then in Corollary 2, we need only assume $A \in O(G)$ and B is a p-subgroup. This version of Corollary 2 is, of course, true for l-groups. Is the intersection of two p-subgroups a p-subgroup? Both of these conjectures seem rather dubious and this probably where the analogy between l-groups and p-groups breaks down.

Proposition 2. 4. *A p-subgroup K of G is a p-group, but a subgroup of G that is a p-group in the induced order need not be a p-subgroup.*

PROOF. If k is an element of a p-subgroup K of G, then $k = a - b$ where a and b are p-disjoint in G and belong to K. Let M be an o-ideal of K that is maximal without a. If $a \in M\sigma$, then $0 \leqq a \leqq m \in M$, so $a \in M$, a contradiction. Thus, $a \notin M\sigma$ so $M\sigma \subseteqq N$, a value of a in G. Hence, $b \in N \cap K \supseteqq M$ and $a \notin N \cap K$, which by the last Lemma is an o-ideal of K. Therefore, $b \in N \cap K = M$. Example (7. 4) establishes the remainder of the proposition.

III. Principal o-ideals and absolute values of an element

For a subset S of G we define $G(S)$ to be the o-ideal generated by S. Then $G(S)$ is the intersection of all o-ideals of G that contain S. If $0 < g \in G$, then

$$G(g) = [\{x \in G \mid 0 \leqq x \leqq ng \text{ for some } n > 0\}]$$

and $G(g)$ is the intersection of all convex subgroups of G that contain g, ([2] p. 207)

Proposition 3. 1. *Suppose $g = a - b \in G$, where a and b are p-disjoint.*

(i) $G(g) = G(a+b) = G(a) + G(b)$.

(ii) *$G(a) \cap G(b) = H(a, b) = H(na, nb)$ for all $n > 0$. Thus, $H(a, b)$ is the intersection of all o-ideals (or convex subgroups) of G that contain a or b.*

(iii) *$x, y \in G^+$ are p-disjoint if and only if x and $y \geqq G(x) \cap G(y)$.*

PROOF. (i) This is clear if $g = 0$. So suppose $g \neq 0$, then there exists $z \in G(g)$ such that $z > g = a - b$ and 0, and so $z + b > a$. By 4. 5 in [2], it follows that $2z \geqq a$. Thus, $a, b \in G(g)$ so $G(a+b) \subseteqq G(g)$. Since $a, b \in G(a+b)$, $g \in G(a+b)$, so we have $G(a+b) \supseteqq G(g)$.

$G(a) + G(b)$ is an o-ideal that contains $a+b$ and any o-ideal containing $a+b$ must contain $G(a)$ and $G(b)$. Therefore, $G(a+b) = G(a) + G(b)$.

(ii) Clearly, $H(a, b) \subseteqq H(na, nb)$. Suppose, by way of contradiction, that $0 \leqq x \in H(na, nb)$ but $x \not\leqq a$. Then there exists a value M of $x - a$ such that $M + x > M + a \geqq M$ and since $M + na \geqq M + x$ we have $M + a > M$. Thus, $M \subseteqq N$ a value of a and $N + x \geqq N + a > N$. But $nb \in N$ so $x \in N$, a contradiction. Therefore, $H(a, b) = H(na, nb)$.

If $0 \leq x \in H(a, b)$, then $x < a$ and $x < b$ and so $x \in G(a) \cap G(b)$. Conversely, if $0 \leq x \in G(a) \cap G(b)$, then $0 \leq x \leq na$ and nb for some $n > 0$ so that $x \in H(na, nb) = H(a, b)$.

(iii) If x and y are p-disjoint and $u \in G(x) \cap G(y)$, then let 0 and $u \leq v \in G(x) \cap G(y) = H(x, y)$. Then $u \leq v \leq x$ and y. Conversely, suppose that x and $y \geq G(x) \cap G(y)$. If $z \leq x$ and y, then there exists $w \in G$ such that 0 and $z \leq w \leq x$ and y so $w \in G(x) \cap G(y)$ and $nz \leq nw \in G(x) \cap G(y) \leq x$ and y. Thus, by Theorem 4. 5 in [2], x and y are p-disjoint.

Theorem 3. 2. *If $g = a - b \in G$ where a and b are p-disjoint, then*

$$\frac{G(g)}{G(a) \cap G(b)} = \frac{G(a) + G(b)}{H(a, b)} \cong \frac{G(a)}{H(a, b)} \boxplus \frac{G(b)}{H(a, b)} \cong \frac{G(g)}{G(a)} \boxplus \frac{G(g)}{G(b)}$$

where $\boxplus$ denotes the cardinal sum.

Proof. $$\frac{G(a)}{H(a, b)} = \frac{G(a)}{G(a) \cap G(b)} \cong \frac{G(b) + G(a)}{G(b)} = \frac{G(g)}{G(b)}$$

by the above proposition and Corollary 2 to the I. H. T., so the first and last parts follow from the above theory

Let $H = H(a, b)$ and for $x \in G(a)$ and $y \in G(b)$ define the map

$$H + x + y \rightarrow (H + x, H + y)$$

$$\text{of } \frac{G(a) + G(b)}{H} \text{ into } \frac{G(a)}{H} \boxplus \frac{G(b)}{H}.$$

If $H + x + y = H + \bar{x} + \bar{y}$ for $\bar{x} \in G(a)$ and $\bar{y} \in G(b)$, then $x - \bar{x} + y - \bar{y} \in H \subseteq G(b)$ and $y - \bar{y} \in G(b)$. Thus, $x - \bar{x} \in G(a) \cap G(b) = H$, and similarly, $y - \bar{y} \in H$. Therefore, the map is an isomorphism of $\dfrac{G(a) + G(b)}{H}$ onto $\dfrac{G(a)}{H} \boxplus \dfrac{G(b)}{H}$.

To complete the proof it suffices to show that $H + y + x \geq H$ implies $H + x \geq H$ and $H + y \geq H$. Now $H + x + y \geq H$ implies there exists $h \in H$ such that $h + x + y \geq 0$ so we may assume $x + y \geq 0$, $x = x_1 - x_2$ and $y = y_1 - y_2$ where x_1, x_2 and y_1, y_2 are p-disjoint pairs in G. Since $G(a)$ and $G(b)$ are o-ideals and $x \in G(a)$, $y \in G(b)$ we have $x_1, x_2 \in G(a)$ and $y_1, y_2 \in G(b)$.

By way of contradiction, suppose $x_2 \notin H$. Then $H \subseteq M$, a value of x_2 and $x_1 \in M$. Now

$$(M + x_2) \wedge (M + y_1) = (M + x_2) \wedge (M + y_2) = M.$$

For if $M + z \leq M + x_2$ and $M + y_1$, then $m_1 + z \leq x_2$ and $m_2 + z \leq y_1$ for some $m_1, m_2 \in M$. Let $m_1, m_2 \geq m \in M$, then there exists $t \in G$ such that 0 and $m + z \leq t \leq x_2$ and y_1. Thus, $t \in G(a) \cap G(b) = H \subseteq M$ and so $M + z \leq M + t = M$. Hence, $(M + x_2) \wedge (M + y_1) = M$ and similarly, $(M + x_2) \wedge (M + y_1) = M$.

If M' is the o-ideal of G that covers M, then M'/M is an archimedean o-group and each positive element in $(G/M)\setminus(M'/M)$ exceeds every element in M'/M ([2], 4. 6). Thus, it follows that $y_1, y_2 \in M$ and so $M+x+y = M-x_2 < M$ which contradicts the fact that $x+y$ is positive. Therefore, $x_2 \in H$ and $H+x = H+x_1 \geqq H$. Similarly, $H+y = H+y_1 \geqq H$. The proof is complete.

Consider $g = a-b \in G$ where a and b are p-disjoint and define the *positive* and *negative* parts and the *absolute value* of g as follows,

$$g^+ = a+H(a, b), \quad g^- = b+H(a, b), \quad |g| = g^+ + g^- = a+b+H(a, b).$$

If G is an l-group, then $H(a, b)=0$ and these agree with the standard definitions in l-groups. In [5] it is shown that $g = x-y$, where x and y are p-disjoint, if and only if $x = a+m$ and $y = b+m$ for some $m \in H(a, b)$. Thus,

$$g^+ = \{x \in G | g = x-y;\ x, y\ p\text{-disjoint}\},$$

$$g^- = \{y \in G | g = x-y;\ x, y\ p\text{-disjoint}\}.$$

In particular,

(i) $g^+ \cup g^- \subseteqq G^+$ and $|g| = g$ if and only if $g \geqq 0$,

(ii) $0 \in g^+$ if and only if $g \leqq 0$, if and only if $g^+ = 0$, if and only if $g = -(g^-)$,

(iii) $0 \in g^-$ if and only if $g \geqq 0$, if and only if $g^- = 0$, if and only if $g = g^+$.

Thus, if $g \neq 0$, then $|g|$ consists of strictly positive elements. We next derive some of the useful properties of g^+, g^- and $|g|$.

(*a*) $\quad (-g)^+ = g^-; \quad (-g)^- = g^+; \quad |-g| = |g|; \quad g-(g^+) = -(g^-)$

Proof. $-g = b-a$ so $(-g)^+ = b+H(a, b) = g^-$ and $(-g)^- = a+H(a, b) = g^+$. $|-g| = (-g)^+ + (-g)^- = g^- + g^+ = |g|$. Also, $g-(g^+) = (a-b)-a+ H(a, b) = -b+H(a, b) = -(g^-)$.

(b) $|g| = \{x+y | g = x-y;\ x, y\ p\text{-disjoint}\}$ if and only if $H(a, b)$ is divisible by 2.

Proof Let $H = H(a, b)$, $A = \{x+y | g = x-y;\ x, y\ p\text{-disjoint}\}$ and assume $|g| = A$. If $h \in H$, then $a+b+h \in |g| = A$ and so $a+b+h = (a+m)+(b+m)$ where $m \in H$. Thus, $h = 2m$ and so H is divisible by 2. Clearly, $|g| = g^+ + g^- \supseteqq A$. If H is divisible by 2 and $z \in |g|$, then $z = a+b+m = a+b+2\bar{m} = a+\bar{m}+b+\bar{m}$ for some $m, \bar{m} \in H$. Thus, $z \in A$.

In the following theory we make use of the fact that g^+, g^- and $|g|$ are cosets. Thus, we define $n|g|$ by coset addition

$$n|g| = |g| + \ldots + |g| = na+nb+H(a, b)$$

and note that $n|g| = \{nx | x \in |g|\}$ if and only if $H(a, b)$ is divisible by n.

(c) $(ng)^+ = n(g^+)$; $(ng)^- = n(g^-)$; $n|g| = |ng|$ for all $n \geqq 0$.

Proof. $ng = na - nb$ where na and nb are p-disjoint and $H(na, nb) = H(a, b)$. Thus, $(ng)^+ = na+H(a, b) = n(a+H(a, b)) = n(g^+)$. Similarly, $(ng)^- = n(g^-)$, $|ng| = (ng)^+ + (ng)^- = n(g^+ + g^-) = n|g|$.

For subsets A and B of G we define $A<B$ if $a<b$ for all $a\in A$, $b\in B$. Then $\leqq$ is a partial order on the familty of all subsets of G, and clearly, $|g|\geqq 0$, and $|g|=0$ if and only if $g=0$.

(d) (*i*) $g^{+}\leqq|g|$, $g^{-}\leqq|g|$.

(ii) If $g\neq 0$ and $0<m<n$ for integers m and n, then $m|g|<n|g|$.

PROOF. (i) We must show $a+H(a,b)\leqq a+b+H(a,b)$. For $x,y\in H(a,b)$, $a+x\leqq a+b+y$ if and only if $x-y\leqq b$, and the latter holds by (iii) of Proposition 3. 1. Thus, $g^{+}\leqq|g|$ and similarly, $g^{-}\leqq|g|$.

(ii) If $x\in m|g|$ and $y\in n|g|$ with $0<m<n$, then $x=ma+mb+h$ and $y=na+$ $+nb+k$ where $h,k\in H(a,b)$. Thus, $y-x=(n-m)(a+b)+k-h\in|(n-m)g|>0$.

(e) For $x,y\in G$ and $n>0$, $|x|\leqq|y|$ if and only if $n|x|\leqq n|y|$.

PROOF. Suppose $|x|=a+b+H(a,b)$, $|y|=u+v+H(a,b)$ where a,b and u,v are pairs of p-disjoint elements and $n>0$. If $|x|=|y|$, then $H(a,b)=H(u,v)$ and so $n|x|=n|y|$. Suppose $|x|<|y|$ and consider $n(a+b)+s\in n|x|=n(a+b)+$ $+H(a,b)$ and $n(u+v)+t\in n|y|$. Let $\bar{s}\in H(a,b)$ such that $\bar{s}\geqq 0$ and s, and $\bar{t}\in H(u,v)$ such that $\bar{t}\leqq 0$ and t. Then,

$$n(a+b)+s\leqq n(a+b)+\bar{s}\leqq n(a+b+\bar{s}),\quad\text{and}\quad n(u+v+\bar{t})\leqq n(u+v)+\bar{t}\leqq$$
$$\leqq n(u+v)+t.$$

Thus, since $a+b+\bar{s}<u+v+\bar{t}$ we have

$$n(a+b)+s\leqq n(a+b+\bar{s})<n(u+v+\bar{t})\leqq n(u+v)+t.$$

Therefore, $n|x|<n|y|$.

On the other hand, if $n|x|=n|y|$, then $H(a,b)=H(u,v)$ and since $G/H(a,b)$ is semiclosed (nX positive implies X is positive for $X\in G/H(a,b)$) we have $|x|=|y|$. Suppose that $n|x|<n|y|$ and consider $a+b+s\in|x|$ and $u+v+t\in|y|$. Then $n(a+b+s)\in n|x|$ and $n(u+v+t)\in n|y|$ so $n(a+b+s)<n(u+v+t)$. Therefore since G is semiclosed, $a+b+s<u+v+t$ so $|x|<|y|$.

(f) If $u\in|g|$, then u and g have the same set of values and if M is a value of u, then $M+u=M+g$ or $M+u=M-g$.

PROOF. If $u\in|g|$, then $u=a+b+h$ where $h\in H(a,b)$. In [2] it is shown that g and $a+b$ have the same set of values and in [5] it is shown that h belongs to each value of $a+b$. Thus, g and u have the same set of values. If M is a value of u, then either $a\in M$ and $M+u=M+b=M-g$ or $b\in M$ and $M+u=M+a=M+g$.

(g) $|g+h|\leqq 2|g|+2|h|$ with equality if and only if $g=h=0$.

PROOF. This is clear if $g+h=0$. So suppose $g+h\neq 0$, $g=a-b$ and $h=x-y$ where a,b and x,y are pairs of p-disjoint elements and consider $u\in|g+h|$ and $v\in 2|g|+2|h|$. Then $v=2(a+b+x+y)+p+q$ where $p\in H(a,b)$ and $q\in H(x,y)$. Let 0 and $p\geqq c\in H(a,b)$ and 0 and $q\geqq d\in H(x,y)$. Then, $v\geqq 2(a+c+b+c)+$ $+2(x+d+y+d)=w$. Let $\bar{a}=a+c$, $\bar{b}=b+c$, $\bar{x}=x+d$, and $\bar{y}=y+d$. Then $g=\bar{a}-\bar{b}$ and $h=\bar{x}-\bar{y}$ with $\bar{a},\bar{b}$ and $\bar{x},\bar{y}$ p-disjoint. It suffices to show $u<w$.

Let M be a value of u, then by (f), M is also a value of $g+h$ and $M+u = M+ +\varepsilon(g+h)$ where $\varepsilon = \pm 1$. Now

$$(1)\quad M\pm\bar{a} \leqq M+2\bar{a},\quad M\pm\bar{x} \leqq M+2\bar{x},$$
$$M\pm\bar{b} \leqq M+2\bar{b},\quad M\pm\bar{y} \leqq M+2\bar{y}.$$

Thus,

$$(2)\quad M+u = M+\varepsilon(g+h) \leqq M+2(\bar{a}+\bar{b}+\bar{x}+\bar{y}) = M+w.$$

If we have equality in (2), then we must have equality in (1), so $\bar{a}$, $\bar{b}$, $\bar{x}$, and $\bar{y}\in M$. Hence, $g+h\in M$, a contradiction. Therefore, $u\neq w$ and $M+u < M+w$.

Let N be a value of $w-u$. If $u\in N$, then $N+u = N < N+w$. If $u\notin N$, then $N\subseteqq M$, a value of u, and by the above, $M+u < M+w$ so we must have $M=N$. Therefore, $N+u < N+w$ for all values N of $w-u$ and so $w>u$.

Proposition 3. 3. *For G, the following are equivalent.*

(1) *G is an l-group.*
(2) *$|g+h| \leqq |g|+|h|$ for all $g, h\in G$.*
(3) *g^+ is a single element for each $g\in G$.*
(4) *If x and y are p-disjoint, then $H(x, y)=0$.*

PROOF. Clearly, (1) implies (2) and since $g^+ = a+H(a, b)$, (3), (4), and (1) are equivalent. Suppose (2) holds, x and y are p-disjoint and $0<c\in H(x, y)$. Let $g=x$ and $h = -y$. Then $g+h = x-y$, $|g+h| = x+y+H(x, y)$ and $|g|+|h| = = x+y$. Since $x+y+c \not\leqq x+y$ we have $|g+h| \not\leqq |g|+|h|$, a contradiction. Thus, $H(x, y)=0$ and (2) implies (4).

Proposition 3. 4. *A subgroup C is an o-ideal of G if and only if $|x|\leqq|c|$ and $c\in C$ imply $x\in C$.*

PROOF. Consider $0 \neq c = a-b\in C$ where a and b are p-disjoint. If $c\in O(G)$, then a and $b\in C$ so $a+b\in C$. Let $x = u-v\in G$ where u and v are p-disjoint, then $u+v+H(u, v) = |x| \leqq |c| = a+b+H(a, b)$. If $|x|<|c|$, then $u+v < a+b < < 2(a+b)$. If $|x|=|c|$, then $u+v+H(u, v) = a+b+H(a, b)$ implies $H(u, v) = = H(a, b)$ and $u+v+h = a+b$ for some $h\in H(a, b)$. Now $-h \leqq a \leqq a+b$ so $u+v \leqq 2(a+b)$. Thus,

$$0\leqq u \quad\text{and}\quad v \leqq u+v \leqq a+b\in C$$

so u, v and $x = u-v\in C$.

Conversely, suppose the condition is satisfied and $c = a-b \leqq a+b = x$. Then, $|x| = a+b \leqq 2a+2b+H(a, b) = |2c|$. Thus, $x\in C$ so C is directed. If $0\leqq y\leqq c\in C$, then $|y|\leqq|c|$ so $y\in C$ and C is convex. Hence, C is an o-ideal.

Proposition 3. 5. $G(g) = G(|g|) = G(g^+)+G(g^-) = \{x\in G\,|\,|x| \leqq n|g|$ *for some* $n>0\}$.

PROOF. Let $g = a-b$ with a and b p-disjoint, then $G(g) = G(a+b) = = G(a)+G(b)$. If $0<x\in G(a)$, then $x\leqq na\in G(g^+)$ for some $n>0$ and so $G(a)\subseteqq \subseteqq G(g^+)$. If $x\in g^+$, then $x = a+m$, $m\in H(a, b)$ and since $H(a, b)\subseteqq G(a)$, we have $x\in G(a)$. Therefore, $G(a)=G(g^+)$ and similarly, $G(b)=G(g^-)$ so $G(g) = G(g^+)+ +G(g^-)$. If $0<x\in G(g)$, then $x \leqq n(a+b)$ and $a+b\in G(|g|)$ so $G(g)\subseteqq (G|g|)$.

If $z \in |g|$, then $z = a+b+m$ where $m \in H(a, b) \subseteq G(a) \subseteq G(a+b)$ so $z \in G(a+b) = G(g)$. Hence, $|g| \subseteq G(g)$ and, since $G(|g|)$ is the o-ideal generated by $|g|$, we have $G(|g|) \subseteq G(g)$.

Let $X = \{x \in G \mid |x| \leqq n|g| \text{ for some } n > 0\}$ and assume $g \neq 0$. Now $ng \in G(g)$ so if $|x| \leqq n|g| = |ng|$, then $x \in G(g)$ by Proposition 3. 4. Thus, $G(g) \supseteq X$. Equality will be established if we can show X is a group, for then X is an o-ideal that contains g and so $X \supseteq G(g)$. If x and $y \in X$, then $|x-y| \leqq 2|x| + 2|-y| = 2|x| + 2|y|$ where $|x|$ and $|y| \leqq n|g|$ for some $n > 0$. Thus, $|x-y| \leqq 4n|g|$ so $x-y \in X$ and X is a group.

Proposition 3. 6. *If S is a subgroup of G and G is divisible by 2, then*

$$T = \{x \in S \mid |y| \leqq |x| \quad \textit{implies} \quad y \in S\}$$

is the largest o-ideal of G contained in S.

The proof is straightforward and will be omitted.

IV. The o-ideal H and prime and lex o-ideals

Let $H = V\{H(a, b) \mid a \text{ and } b \text{ are } p\text{-disjoint in } G\}$.

Theorem 4. 1. *For $K \in O(G)$, the following are equivalent.*

(1) *G/K is an l-group.*

(2) *$K \supseteq H$.*

Proof. If G/K is an l-group and a and b are p-disjoint in G, then $K+a$ and $K+b$ are p-disjoint in G/K and $(K+a) \wedge (K+b) = K$. If $0 \leqq m \in H(a, b)$ then $K \leqq K+m \leqq K+a$ and $K+b$ so $K+m = K$. Thus, $m \in K$ and $H(a, b) \subseteq K$. Therefore, $K \supseteq H$.

Conversely, suppose that $K \supseteq H$ and let $X \in G/K$. Then $X = K+g = (K+a) - (K+b)$ where by Proposition 2. 2 we may assume a and b are p-disjoint in G. By the same proposition,

$$H(K+a, K+b) = (K + H(a, b))/K = K.$$

Thus, $(K+a) \wedge (K+b) = K$ and G/K is an l-group.

Corollary 1. *G is an l-group if and only if $H = 0$.*

Corollary 2. *The following are equivalent.*

(a) *There exists an o-ideal K of G such that both K and G/K are l-groups.*
(b) *H is an l-group.*

Proof. If (a) is true, then $K \supseteq H$ and so H is an l-ideal of K and, in particular, H is an l-group. The converse is trivial.

Corollary 3. *If $\{G_\delta \mid \delta \in \Delta\}$ is a set of o-ideals of G such that each G/G_δ is an l-group, then $G/\left(\bigcap \{G_\delta \mid \delta \in \Delta\}\right)$ is an l-group.*

Proof. Each $G_\delta \supseteq H$ so $\bigcap \{G_\delta \mid \delta \in \Delta\} \supseteq H$.

Definition. An o-ideal M of G is *prime* if G/M is an o-group.

Proposition 4. 2. $H = \bigcap \{M | M$ is a prime o-ideal of $G\}$.

PROOF. If M is a prime o-ideal of G, then G/M is an o-group, so by Theorem 4. 1, $M \supseteqq H$. Therefore, $H \subseteqq \bigcap \{M | M$ is a prime o-ideal of $G\}$. If $0 < g \in G \setminus H$, then since G/H is an l-group, there exists a prime l-ideal $\mathcal{M}$ of G/H that does not contain $H+g$ (any l-ideal maximal without $H+g$ will do). Now $\mathcal{M} = M/H$ where M is an o-ideal of G and $G/M \cong (G/M)/(M/H)$, an o-group. Thus, M is a prime o-ideal of G and $g \notin M$. Hence, we have shown, $g \notin H$ implies $g \notin \bigcap \{M | M$ is a prime o-ideal of $G\}$. Therefore, it follows that $H = \bigcap \{M | M$ is a prime o-ideal of $G\}$.

Proposition 4. 3. *For $M \in O(G)$, the following are equivalent.*

(1) *M is prime.*

(2) *The o-ideals of G that contain M form a chain.*

(3) *If a and b are p-disjoint in G, then $a \in M$ or $b \in M$.*

PROOF. (1) implies (2). There is a one-to-one, inclusion preserving correspondence between the o-ideals of G that contain M and the o-ideals of G/M. Clearly, the latter form a chain.

(2) implies (3). If $a \notin M$ and $b \notin M$, then there exists a value A of a such that $A \supseteqq M$ and a value B of b such that $B \supseteqq M$. But then A and B are comparable, which contradicts the fact that a and b are p-disjoint.

(3) implies (1). If $M+g \in G/M$, then $g = a-b$ where a and b are p-disjoint. Either $b \in M$ and $M+g = M+a \geqq M$, or $a \in M$ and $M+g = M-b \leqq M$. Therefore, G/M is an o-group.

Remark. Each subgroup M of G that satisfies (3) is clearly a p-subgroup and any subgroup that contains a prime o-ideal satisfies (3). A subgroup M of an l-group satisfies (3) if and only if M contains a prime l-ideal, but we have been unable to prove this for p-groups.

Corollary 1. *If $\ldots \supset \ldots G_\delta \supset \ldots$ is a chain of prime o-ideals of G, then $\bigcap G_\delta$ is a prime o-ideal. In particular, each prime o-ideal contains a minimal prime o-ideal.*

PROOF. This is an immediate consequence of (3).

Definition. An *l-ideal* of G is an o-ideal which is also a lattice with respect to the induced partial order. Example (7. 3) shows that the join of two l-ideals of G need not be an l-ideal.

Proposition 4. 4. *If A is an l-ideal of G and $a, b \in A$, then $a \wedge_A b$ is the g. l. b. of a and b in G, and $a \vee_A b$ is the l. u. b. of a and b in G. Thus, the set of l-ideals of G is closed with respect to intersections and joins of chains.*

PROOF. Let A be an l-ideal of G. If $x \leqq a$ and b, then since G is Riesz, there is $y \in G$ such that $x \leqq y \leqq a$ and $(a \wedge_A b) \leqq y \leqq b$. Since A is convex, $y \in A$ so $a \wedge_A b = y$. Thus, $x \leqq a \wedge_A b$ and so $a \wedge_A b$ is the g. l. b. of a and b in G. A dual argument establishes the remainder of the proposition.

Definition. An o-ideal C of G is *lex* if $x \in G^+ \setminus C$ implies $x > C$.

It follows at once that an o-ideal C of G is lex if and only if each strictly positive element in G/C consists of positive elements. Let C_1 and C_2 be two lex o-ideals of G and suppose that $0 < g \in C_1 \setminus C_2$. Then $g > C_2 > -g$ and hence, $C_2 \subseteqq C_1$. Therefore, the set $\mathscr{L}$ of all lex o-ideals of G form a chain (with 0 as the least element and G as the largest element).

$\mathscr{L}$ is closed with respect to joins and intersections. For let $\mathscr{I}$ be a subset of $\mathscr{L}$ and consider $J = \bigcup \mathscr{I}$ and $K = \bigcap \mathscr{I}$. If $g \in G^+ \setminus J$, then $g \in G^+ \setminus T$ for all $T \in \mathscr{I}$ and so $g > T$ for all $T \in \mathscr{I}$. Hence, $g > J$. If $g \in G^+ \setminus K$, then $g \notin T$ for some $T \in \mathscr{I}$ and $g > T \supseteqq K$.

Since the join of a chain of l-ideals is an l-ideal, there exists a largest lex o-ideal L of G which is also an l-ideal. We next show that there exists a smallest lex o-ideal S such that G/S is an l-group. For let $\mathscr{I}$ be the collection of all lex o-ideals T of G such that G/T is an l-group, and let $S = \bigcap \mathscr{I}$. Then S is a lex o-ideal and by Corollary 3 to Theorem 4. 1, G/S is an l-group.

Note that the following are equivalent: G is an l-group; $H=0$; $S=0$; $L=G$; G/L is an o-group. For if G/L is an o-group, then G is a lexicographic extension of the l-group L by the o-group G/L and so, G is an l-group.

Proposition 4. 5. *If G is an l-group, then $0 = S = H \subseteqq L = G$. If G is not an l-group, then $L \subseteqq H \subseteqq S$. If S is not prime, then $S = H$.*

Proof. We have established the part when G is an l-group. So suppose G is not an l-group. Then G/L is not an o-group so there exist strictly positive elements X and Y in G/L that are p-disjoint. By Proposition 2. 2, $X = L + x$ and $Y = L + y$ where x and y are p-disjoint in G. Since $x, y \in G^+ \setminus L$, we have $x, y > L$ and so $L \subseteqq H(x, y) \subseteqq H$. Since G/S is an l-group, Proposition 4. 1 establishes $H \subseteqq S$. If S is not prime, then by Proposition 4. 3, there exists a and b p-disjoint in G such that $a \notin S$ and $b \notin S$. Thus, $a, b > S$ so $S \subseteqq H(a, b) \subseteqq H$.

Proposition 4. 6. *For G, the following are equivalent.*

(1) *G is a lexicographic extension of an l-group by an l-group.*
(2) *G/L is an l-group (or equivalently $H \subseteqq L$).*
(3) *S is an l-group.*
(4) *$S \subseteqq L$.*
(5) *$S = H$ and H is an l-group.*
(6) *G is a lexicographic extension of the l-group H by the l-group G/H.*

Proof. Clearly, (1), (2), and (3) are equivalent, (2) implies (4), (5) implies (6) and (6) implies (1).

(4) implies (3). S is an l-ideal of L and hence, S is an l-group.

(3) implies (5). If S is an l-group but not prime, then $S = H$ by the last proposition. If S is prime, then G is a lexicographic extension of S by the o-group G/S and so G is an l-group. Hence, $H = S = 0$.

V. The p-group $V(\Delta, H_\delta)$

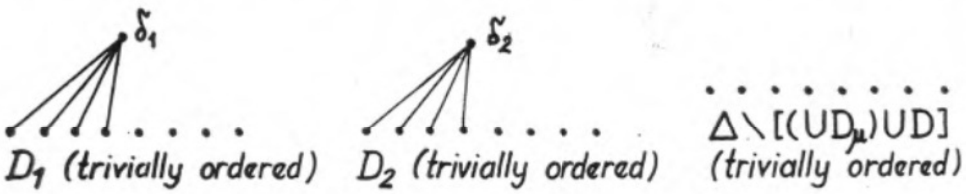
Let H_δ be a po-group for each δ in a po-set Δ and let $V = V(\Delta, H_\delta)$ be the set of all Δ-vectors $v = (\dots, v_\delta, \dots)$ where $v_\delta \in H_\delta$, for which the support $S(v) = \{\delta \in \Delta \mid v_\delta \neq 0\}$, contains no infinite ascending chains. Define $0 \neq v \in V$ to be *positive* if $v_\delta > 0$ for each maximal element $\delta \in S(v)$ (that is, if each *maximal component* is positive). Then V is a *po*-group [1] and it can be shown that V is a *p*-group if and only if each H_δ is a *p*-group. If each H_δ is an *o*-group, then V is a *p*-group (the proof of Theorem 4. 8 in [2] establishes this) and in [1] it is shown that V is an *l*-group if and only if Δ is a *root sytem* (that is, for each $\gamma \in \Delta$, $\{\delta \in \Delta | \delta \geqq \gamma\}$ is a chain).

Theorem 5. 1. *If each H_δ is an o-group, then $V = V(\Delta, H_\delta)$ is a p-group and V^+ is the union of lattice cones.*

Proof. Let H_δ be an *o*-group for each $\delta \in \Delta$, then V is a *p*-group by the above. Let D be a trivially ordered subset of Δ. Well order D as $\delta_1, \delta_2, \dots \delta_\mu, \delta_{\mu+1}, \dots$ and for each μ, let

$$D_\mu = \{\delta \in \Delta | \delta < \delta_\mu \text{ and } \delta \nless \delta_\nu \text{ for any } \nu < \mu\}.$$

We assign a new partial order to Δ by defining α to be greater than β if $\alpha = \delta_\mu$ for some $\delta_\mu \in D$ and $\beta \in D_\mu$. Let Δ_D be the set Δ with this new partial order. Then clearly, Δ_D is a root system and this partial order is weaker than the given partial order. The following is a "picture" of the new partial order. The *l*-group



$V_D = V(\Delta_D, H_\delta)$ is the large direct sum of the H_δ and V is a subgroup of V_D. If $v \in V$, then $S(v \vee_D 0) \subseteq S(v)$ so $(v \vee_D 0) \in V$ and hence, V is an *l*-subgroup of V_D. Next, $V^+ \supseteq V \cap V_D^+$ and so $V^+ \supseteq \cup (V \cap V_D^+)$ for all trivially ordered subsets D of Δ.
Now consider $v \in V^+$ and let D be the set of maximal elements in $S(v)$. Then v has exactly the same maximal components in V_D so $v \in V \cap V_D^+$. Therefore, $V^+ = \cup (V \cap V_D^+)$ for all trivially ordered subsets D of V.

Corollary 1. *If Δ contains only a finite number of trivially ordered subsets, then V^+ is the union of a finite number of lattice cones.*

Corollary 2. *Each p-group G is a p-subgroup of a p-group V for which V^+ is the union of lattice cones.*

Proof. By Theorem 4. 10 in [2], G is *p*-isomorphic to a *p*-subgroup of $V(\Delta, R_\delta)$ where each $R_\delta = R$.

An obvious question is whether or not G^+ is the union of lattice cones. A partial answer makes use of the following construction. For elements α and β in a *po*-set Δ we define $\alpha \sim \beta$ if,

(1) α and β are comparable,
(2) the closed interval determined by α and β is a chain,
(3) δ is comparable to α if and only if δ is comparable to β, for all $\delta \in \Delta$.

I) $\sim$ is an equivalence relation.

Let $\tilde{\alpha}$ denote the equivalence class that contains α, and define

$$\tilde{\alpha} < \tilde{\beta} \text{ if } \tilde{\alpha} \neq \tilde{\beta} \text{ and } \alpha < \beta.$$

II) The set $\Lambda = \{\tilde{\alpha} | \alpha \in \Delta\}$ is partially ordered with respect to this definition and Λ is a root system f and onlyi if Δ is a root system.

III) Λ is finite if and only if Δc ontains only a finite number of maximal chains.

The proofs of I—III are reasonably straightforward and we shall omit them.

Now as in [2], let $\Delta = \Delta(G)$ be an index set for the set of all pairs (G^δ, G_δ) of o-ideals of G such that G_δ is maximal without some $g \in G$, and G^δ covers G_δ. Each G^δ/G_δ is o-isomorphic to a subgroup of the additive group R of real numbers with the natural order. Let Λ be as above and for each $\lambda \in \Lambda$, define

$$H^\lambda = \bigcup_{\alpha \in \lambda} G^\alpha \qquad H_\lambda = \bigcap_{\alpha \in \lambda} G_\alpha.$$

Then each $H^\lambda \diagdown H_\lambda$ is an o-group. For if $x \in H^\lambda \diagdown H_\lambda$, then $x \in G^\alpha \diagdown G_\alpha$ for some $\alpha \in \lambda$ and $x = a - b$ where a and b are p-disjoint in G. Since α is a value of x, it must be a value of a or of b. If $a \in G^\alpha \diagdown G_\alpha$, then $b \in G_\alpha$ and, since no value of b is comparable to α, $b \in H_\lambda$. Thus, $H_\lambda + x = H_\lambda + a > H_\lambda$, and if $b \in G^\alpha \diagdown G_\alpha$, then $H_\lambda + x = H_\lambda - b < H_\lambda$.

For $\mu, \nu \in \Lambda$, $\mu < \nu$ if and only if $H^\mu \subseteqq H_\nu$ and also

(1) $0 \neq g \in G$ implies $g \in H^\lambda \diagdown H_\lambda$ for some $\lambda \in \Lambda$.

(2) $g \notin H^\lambda$ implies $g \in H^\mu \diagdown H_\mu$ for some $\lambda < \mu \in \Lambda$.

Now if G is divisible, then it can be shown that there exists a p-isomorphism of G into $V(\Lambda, H^\lambda \diagdown H_\lambda)$, but we make no use of this result.

Theorem 5. 2. *If G is divisible and Δ contains only a finite number of maximal chains, then Λ is finite and $G \cong V(\Lambda, H^\lambda/H_\lambda)$ where each H^λ/H_λ is an o-group. In particular, G^+ is the union of a finite number of lattice cones.*

PROOF. By III, Λ is a finite po-set and by (3. 1) in [2] each o-ideal is a pure subgroup of G and hence, divisible. Thus, each H^λ is a direct summand of G so $G = H^\lambda \oplus D^\lambda$. Consider $g \in G$ and $\lambda \in \Lambda$, we can write $g = g_\lambda + d_\lambda$ where $g_\lambda \in H^\lambda$ and $d_\lambda \in D^\lambda$. Define

$$g\tau = (\ldots, H_\lambda + g_\lambda, \ldots).$$

It is clear that τ is a homomorphism of G into V and because of (1), τ is an one-to-one. It is easy to check that the following are equivalent: $g > 0$; $H_\lambda + g > H_\lambda$ for all $\lambda \in \Lambda$ such that $g \in H^\lambda \diagdown H_\lambda$; each maximal component of $g\tau$ is positive. Therefore, τ is an o-isomorphism of G into $G\tau$.

To prove τ is onto, consider $0 \neq v \in V$. To show there is a $y \in G$ such that $y\tau = v$ we will use induction on the cardinality of

$$T_v = \{\lambda \in \Lambda | \lambda \leqq \mu \quad \text{for some} \quad \mu \in S(v)\}.$$

Let $H_\lambda + g$ be a maximal component of v and suppose (*) there exists $h \in G$ such that λ is the only value of h in Λ and $H_\lambda + h = H_\lambda + g$. Let $s = v - h\tau$. Then clearly T_s is a proper subset of T_v and so by induction there is $x \in G$ such that $x\tau = s$. Hence, $(x + h)\tau = s + h\tau = v$ and τ is onto.

We may, without loss of generality, assume $g > 0$. Let $\alpha_1, \ldots, \alpha_n$ be the values of g in Δ. Note that Δ having only a finite number of maximal chains implies that g is finite valued. Then $\lambda_i = \tilde{\alpha}_i$, $1 \leqq i \leqq n$ are the values of g in Λ. By (4. 10) in [2], for each $i = 2, \ldots, n$ we may select $0 > k_i \in G$ with α_i as its only value and such that

$G_{\alpha_i}+g+k_i < G_{\alpha_i}$. Let $k = k_2 + \ldots + k_n$. Then $0 > k \in H_{\lambda_1} \setminus H_{\lambda_i}$ and $H_{\lambda_1}+g+k < H_{\lambda_i}$ for $i=2, \ldots, n$.

Suppose $g+k \in H^\lambda \setminus H_\lambda$ and $H_\lambda+g+k > H_\lambda$. Then $g \notin H_\lambda$; for otherwise $H_\lambda+g+k = H_\lambda+k \leqq H_\lambda$. If $g \notin H^\lambda$, then $H^\lambda \subseteqq H_{\lambda_j}$ for some $j=1, 2, \ldots, n$ and since $g+k \in H^\lambda \subseteqq H_{\lambda_j}$ and $g \notin H_{\lambda_j}$, it follows that $k \notin H_{\lambda_j}$. But then $H_{\lambda_j}+g+k < H_{\lambda_j}$, a contradiction. Therefore, $g \in H^\lambda \setminus H_\lambda$ and hence, $\lambda = \lambda_1$. Thus, $g+k$ has exactly one positive value λ_1. Now $g+k = a-b$ where a and b are p-disjoint and it follows that the values of a are the positive values of $g+k$, namely λ_1. Moreover, $H_{\lambda_1}+g = H_{\lambda_1}+g+k = H_{\lambda_1}+a-b = H_{\lambda_1}+a$. This establishes (*) and the proof is complete.

Remark. If G is an l-group, then the following are equivalent: Δ contains only a finite number of maximal chains; G has a finite basis; G has only a finite number of minimal primes. Thus, the last theorem is the structure theorem for an abelian l-group with a finite basis. See [1, p. 161].

VI. p-groups which are o-homomorphic images of l-groups

This section is devoted to proving the following result.

Theorem 6. 1. *If G is a divisible p-group and $\Delta=\Delta(G)$ contains only a finite number of maximal chains, then there exists an l-group H with a finite basis and a trivially ordered subgroup C of H such that G and H/C are o-isomorphic.*

In order to prove this, we first derive two lemmas. Suppose that Γ and Λ are po-sets and θ is a map of Γ onto Λ such that

(i) $\alpha<\beta$ implies $\alpha\theta<\beta\theta$

(ii) $\alpha\theta<\beta\theta$ implies $\bar{\gamma}\theta=\bar{\alpha}\theta$ for some $\gamma<\beta$

where $\bar{\gamma}=\{\delta\in\Gamma \mid \delta\leqq\gamma\}$. Let $H=\Sigma(\Gamma, H_\gamma)$ and $G=\Sigma(\Lambda, G_\lambda)$ where the H_γ and G_λ are o-groups such that $H_\gamma=G_\lambda$ if $\gamma\theta=\lambda$, and $\Sigma(\Gamma, H_\gamma)(\Sigma(\Lambda, G)_\lambda)$ is the subgroup of elements with finite support in $V(\Gamma, H_\gamma)(V(\Lambda, G_\lambda))$. For $h\in H$, we define $h\pi\in G$ as follows

$$(h\pi)_\lambda = \sum_{\gamma\theta=\lambda} h_\gamma .$$

Lemma A. *π is an o-homomorphism of H onto G and so $G\cong H/K(\pi)$. Moreover, $K(\pi)$ is trivially ordered.*

PROOF. $(g\pi)_\lambda+(h\pi)_\lambda = \sum_{\gamma\theta=\lambda} g_\gamma + \sum_{\gamma\theta=\lambda} h_\gamma = \sum_{\gamma\theta=\lambda}(g+h)_\gamma = ((g+h)\pi)_\lambda$

and hence, π is a homomorphism of H onto G. Consider $0\neq h\in H$ with maximal components $h_{\gamma_1}, \ldots, h_{\gamma_n}$ and let $\{\gamma_1, \ldots, \gamma_n\}=S$.

(1) If $\gamma_i\theta$ is maximal in $S\theta$, then $(h\pi)_\lambda=0$ for all $\lambda>\gamma_i\theta$ and

$$(h\pi)_{\gamma_i\theta} = \sum_{\gamma_j\theta=\gamma_i\theta} h_{\gamma_j} .$$

For if $\lambda>\gamma_i\theta$ and $(h\pi)_\lambda\neq0$, then there exists $\gamma\in\Gamma$ such that $h_\gamma\neq0$ and $\gamma\theta=\lambda$. But then $\gamma \leqq \gamma_j$ for some j and $\gamma_i\theta < \lambda = \gamma\theta \leqq \gamma_j\theta$ which contradicts the maximality of $\gamma_i\theta$. Therefore, $(h\pi)_\lambda = 0$ for all $\lambda > \gamma_i\theta$. Now, by definition,

$$(h\pi)_{\gamma_i\theta} = \sum_{\gamma\theta=\gamma_i\theta} h_\gamma .$$

Suppose that $h_\gamma \neq 0$ and $\gamma\theta = \gamma_i\theta$. Then $\gamma \leqq \gamma_j$ for some j, and if $\gamma < \gamma_j$, then $\gamma_i\theta = \gamma\theta < \gamma_j\theta$ which again contradicts the maximality of $\gamma_i\theta$. Thus, $\gamma = \gamma_j$ and (1) holds.

(2) If $0 < h \in H$, then $0 < h\pi$. In particular $K(\pi)$ is trivially ordered. For if $\gamma_i\theta$ is maximal in $S\theta$, then by (1), $(h\pi)_{\gamma_i\theta}$ is a positive maximal component of $h\pi$ and so it suffices to show that these are the only maximal components of $h\pi$. If $(h\pi)_\lambda$ is a maximal component of $h\pi$, then there exists $\gamma \in \Gamma$ such that $h_\gamma \neq 0$ and $\gamma\theta = \lambda$. Now $\gamma \leqq \gamma_j$ for some j and so $\lambda = \gamma\theta \leqq \gamma_j\theta \leqq \gamma_k\theta$ where $\gamma_k\theta$ is maximal in $S\theta$. Thus, $(h\pi)_{\gamma_k\theta}$ is a maximal component of $h\pi$ and so, $\lambda = \gamma_k\theta$.

(3) If $h\pi > 0$, then there exists $0 < x \in H$ such that $x\pi = h\pi$. We use induction on the number s of elements in the support of h. The result is clear if $s = 1$.

Case I. There exists a maximal element $\gamma_i\theta$ in $S\theta$ for which the summation $c = \sum_{\gamma_j\theta=\gamma_i\theta} h_{\gamma_j}$ contains at least two terms. In h replace h_{γ_i} by c and replace each of the other h_{γ_j} in this summation by 0. This defines $k \in H$ such that $k\pi = h\pi$ and the number of elements in the support of h is less than s. Thus, by induction, there exists $0 < x \in H$ such that $x\pi = k\pi = h\pi$.

Case II. $\gamma_i\theta$ maximal in $S\theta$ implies $(h\pi)_{\gamma_i\theta} = h_{\gamma_i} > 0$. We may assume h is not positive and hence, there exists a maximal component $h_{\gamma_j} < 0$. Since $h\pi > 0$, $\gamma_j\theta < \gamma_i\theta$ where $\gamma_i\theta$ is maximal in $S\theta$. By property (ii), there exists $\gamma < \gamma_i$ such that $\bar{\gamma}\theta = \bar{\gamma}_j\theta$. For each element $v \in \bar{\gamma}_j\theta$, pick an element $\gamma_v \in \bar{\gamma}$ such that $\gamma_v\theta = v$ and define $k \in H$ as

$$k_{\gamma_v} = \sum_{\substack{\delta\theta = v \\ \delta \in \bar{\gamma}_j}} h_\delta .$$

and all other components of k are zero. Now replace all the $h_\alpha \neq 0$, $\alpha \in \bar{\gamma}_j$ by zero and add this result of h to k. This gives an element $t \in H$ with one less negative maximal component than h and such that $t\pi = h\pi$. We proceed in this way to get $0 < x \in H$ such that $x\pi = h\pi$. This completes the proof.

Lemma B. *If Λ is a finite po-set, then there exists a finite root system Γ and a mapping θ of Γ onto Λ such that*

(i) $\alpha < \beta$ *implies* $\alpha\theta < \beta\theta$
(ii) $\alpha\theta < \beta\theta$ *implies* $\bar{\gamma}\theta = \bar{\alpha}\theta$ *for some* $\gamma < \beta$

Proof. Call $\lambda \in \Lambda$ a *branch point* if there exists $\mu \| v$ in Λ such that $\lambda < \mu$ and v, and no element of Λ occurs between λ and μ or between λ and v (that is, μ and v *cover* λ).

Let λ be a minimal branch point. Select two o-isomorphic copies $\bar{\lambda}_1$ and $\bar{\lambda}_2$ of $\bar{\lambda}$ and let ψ_i map $\bar{\lambda}_i$ o-isomorphically onto $\bar{\lambda}$, $i = 1, 2$. Let $\Lambda_1 = (\Lambda \setminus \bar{\lambda}) \cup \bar{\lambda}_1 \cup \bar{\lambda}_2$. We use the natural partial order on each of the three parts of Λ_1, and define

$$\lambda_1 < \mu,\ v_1,\ v_2,\ \ldots,\ \gamma_n, \quad \lambda_2 < v,\ v_1,\ v_2,\ \ldots,\ v_n$$

where $\mu, \nu, \nu_1, \nu_2, \ldots, \nu_n$ are all the elements in Λ which cover λ. Let θ_1 be the map on Λ_1 defined as

$$\theta_1 = \begin{cases} \text{the identity on } \Lambda \setminus \bar{\lambda} \\ \Psi_1 \text{ on } \lambda_1 \\ \Psi_2 \text{ on } \lambda_2 \end{cases}$$

A routine argument establishes that θ_1 satisfies (i) and (ii). Now Λ_1 has one less branch point than Λ. Thus, after a finite number of steps we obtain a (finite) root system Γ and the desired mapping θ.

Proof of Theorem 6. 1. By Theorem 4. 2, we may assume that $G = V(\Lambda, G_\lambda)$ where the G_λ are o-groups and Λ is finite. By Lemma B, there exists a finite root system Γ and a mapping θ of Γ onto Λ that satisfies (i) and (ii). Let H by the l-group $V(\Gamma, H_\gamma)$ where $H_\gamma = G_\lambda$ if $\gamma\theta = \lambda$. Then by Lemma A, there is an o-homomorphism π of H onto G with $K(\pi)$ trivially ordered. Clearly H has a finite basis and $G \cong H/K(\pi)$.

It follows from Theorems 5. 2 and 6. 1 that if G is a finite dimensional real p-space, then G^+ is the union of vector lattice cones, and there exists a finite dimensional vector lattice H with a trivially ordered subspace C such that G and H/C are o-isomorphic.

VII. Examples

(7. 1) We first give a method of constructing p-groups from l-groups. Suppose that $0 = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\beta = H$ is a well ordered chain of l-ideals of an l-group H where, $H_\alpha = \bigcup H_\gamma$ for all $\gamma < \alpha$ if α is a limit ordinal. Then for each $0 \neq h \in H$ there exists an α such that $h \in H_{\alpha+1} \setminus H_\alpha$. Define $h \in H$ to be positive if $h = 0$ or $h \in H_{\alpha+1} \setminus H_\alpha$ and $H_\alpha + h > H_\alpha$. We denote the original order of H by $<$ and the new order by $\lhd$.

Proposition 7. 1. *$\lhd$ is a partial order that extends the given order $<$. Each l-ideal N of $(H, <)$ such that $H_\alpha \subseteq N \subseteq H_{\alpha+1}$ for some α is an o-ideal of $(H, \lhd)$. If $h \in H_{\alpha+1} \setminus H_\alpha$, the then values of h in $(H, \lhd)$ are the values of h in $(H, <)$ between H_α and $H_{\alpha+1}$. Each $H_{\alpha+1}$ is a lexicographic extension of the p-group H_α by the l-group $H_{\alpha+1}/H_\alpha$.*

The proof is straightforward but long; so we omit it.

(7. 2) An example of a finite dimensional vector lattice H with a trivially ordered (hence convex) subspace C such that H/C is not a Riesz group, and hence, not a p-group.

Let $H = R \boxplus R \boxplus R \boxplus R$ and $C = \{(x, -x, x, -x) | x \in R\}$. Each coset in H/C has a representation $(0, x, y, z) + C$ and the following are equivalent.

(i) $(0, x, y, z) + C \geqq C$,

(ii) $(b, x-b, y+b, z-b) \geqq 0$ for some $b \in R$,

(iii) $b \geqq 0, x \geqq b, y \geqq -b, z \geqq b$ for some $b \in R$,

(iv) $x \geqq 0, z \geqq 0$, and $y \geqq -\min\{x, z\}$ (i.e let $b = \min\{x, z\}$). Thus, it follows that

$$\left.\begin{matrix} (0, 0, -1, 1) + C \\ C \end{matrix}\right\} \leqq \left\{\begin{matrix} (0, 0, 0, 1) + C \\ (0, 1, -1, 1) + C \end{matrix}\right.$$

where $(0, 0, 0, 1)+C$ and $(0, 1, -1, 1)+C$ are not comparable. We show

$$\left.\begin{matrix}(0, 0, -1, 1)+C\\ C\end{matrix}\right\} \leqq (0, p, q, r)+C \leqq (0, 0, 0, 1)+C$$

implies $(0, p, q, r)=(0, 0, 0, 1)$ and so G is not a Riesz group.

Now,

$C \leqq (0, p, q, r)+C$ implies $p \geqq 0$, $q \geqq -\min\{p, r\}$

$C \leqq (0, p, q+1, r-1)+C$ implies $r-1 \geqq 0$

$C \leqq (0, -p, -q, 1-r)+C$ implies $-p \geqq 0$, $1-r \geqq 0$, $-q \geqq -\min\{-p, 1-r\}$.

Thus, $p=0$, $r=1$ and so $q=0$.

A simpler example where C is not a subspace is the following. $G = R \boxplus R$, $C = \{(x, -x) | x \text{ is rational}\}$. The following are equivalent.

(i) $C+(a, b) \geqq C$,

(ii) $(a+q, b-q) \geqq 0$, for some $q \in Q$ (=the set of all rational numbers),

(iii) $b \geqq q \geqq -a$, for some $q \in Q$,

(iv) $b = -a \in Q$ or $a+b > 0$.

It follows that $\mathscr{D} = \{C+(a, -a) | a \in R\}$ is a trivially ordered subgroup of $\mathscr{G} = G/C$ and $\mathscr{G}$ is a lexicographic extension of $\mathscr{D}$ by the o-group $\mathscr{G}/\mathscr{D}$. Thus, $\mathscr{G}$ is o-simple and hence, not a p-group. However, $\mathscr{G}$ is a Riesz group.

(7. 3) $G = R \oplus R \oplus R$ with (a, b, c) positive if $a>0$ and $b \geqq 0$, or $a \geqq 0$ and $b>0$, or $a=b=0$ and $c \geqq 0$. Then G is a p-group and $P_1 = \{0\} \oplus R \oplus R$ and $P_2 = R \oplus \{0\} \oplus R$ are prime l-ideals of G, but $P_1 \cup P_2 = G$ is not an l-ideal.

(7. 4) An example of an l-group H with a subgroup S that is an l-group with respect to the induced partial order, but not an l-subgroup.

Let $0 \neq K$ be an abelian l-group; let $H = K \boxplus K \boxplus K$ and

$$S = \{(x, y, x+y) | x, y \in K\} \cong K \boxplus K.$$

For $0<k \in K$, $(2k, -k, k) \vee \theta = (2k, 0, k) \notin S$ where θ denotes the identity of S. Thus, S is not an l-subgroup of H but $(x, y, z) \vee_S \theta = (x \vee 0, y \vee 0, (x \vee 0) + (y \vee 0)) \in S$ and S is a lattice in the induced partial order.

(7. 5) Let $S = \{\varepsilon_\delta | \delta \in \Delta\}$ be a basis for the real vector space H. Assign a partial order to S (or equivalently to Δ) and consider $h = h_1\varepsilon_{\delta_1} + \cdots + h_n\varepsilon_{\delta_n}$ in H. Define h_i to be a maximal component of h if $h_i \neq 0$ and $h_j = 0$ for all $\delta_j > \delta_i$, and define h to be positive if each maximal component of h is positive. Then H is a real p-space and we say S is an *order determining basis* for H. Note that

$$H \cong \sum(\Delta, R_\delta).$$

Conversely, suppose H is a real po-vector space and S is a basis of positive elements such that

(*i*) $\alpha \neq \beta$ implies $R^+\varepsilon_\alpha > \varepsilon_\beta$ or $R^+\varepsilon_\alpha < \varepsilon_\beta$ or $R^+\varepsilon_\alpha \| \varepsilon_\beta$,

(*ii*) $\varepsilon_{\delta_1}, \ldots, \varepsilon_{\delta_n} \| \varepsilon_\delta$ implies $(x_1\varepsilon_{\delta_1} + \ldots + x_n\varepsilon_{\delta_n}) \| \varepsilon_\delta$ for all $0 < x_i \in R$.

Then it can be shown that H is a p-space and S is an order determining basis. We conclude by listing the following.

Open Questions

(1) If $P_1, \ldots, P_n$ are lattice cones for a group H and $P = P_1 \cup \cdots \cup P_n$ is a cone for H, then is (H, P) a p-group?

(2) Is each p-group the homomorphic image of an l-group?

Added in proof: The Answer to questiont (1) above is, no. J. JAKUBIK [5] answers some of the other open questions. For a p-group G he has proven the following results.

A subgroup M of G contains a prime o-ideal if and only if a and b p-disjoint in G implies $a \in M$ or $b \in M$.

If A is an o-ideal and B a p-subgroup of G then $A + B$ is a p-subgroup of G.

The intersection of p-subgroups need not be a p-subgroup.

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