

A note on almost-Dedekind domains

By ELBERT M. PIRTLE, JR. (Kansas City, Mo)

1. Introduction

In this section we state the necessary background results from [1]. In the remainder of this paper, R will denote a commutative integral domain with identity and quotient field K . $I(R)$ will denote the collection of non-zero fractionary ideals of R . A fractionary ideal of the form Rx , $x \neq 0$, $x \in K$, is called a principal fractionary ideal.

A relation $<$ is defined on $I(R)$ as follows: $A < B$ iff every principal fractionary ideal of R which contains A also contains B . The relation $<$ is reflexive and transitive. If we define $\equiv$ on $I(R)$ by $A \equiv B$ iff $A < B$, and $B < A$, for $A, B \in I(R)$, then $\equiv$ is an equivalence relation on $I(R)$. For $A \in I(R)$, $\text{div}_R(A)$ denotes the equivalence class of A with respect to $\equiv$ and is called the divisor of A ; $\mathcal{D}(R)$ denotes the set of all such equivalence classes.

For $A \in I(R)$, we put $\tilde{A} = \bigcap_{A \subseteq Rx} Rx$, $x \in K$, $x \neq 0$. A fractionary ideal B of R is said to be divisorial if $\tilde{B} = B$. It follows that for $A \in I(R)$, $\text{div}_R(A) = \text{div}_R(\tilde{A})$ and that $\tilde{A}$ is the unique divisorial fractionary ideal belonging to $\text{div}_R(A)$. It also follows from the definition that $\widetilde{\tilde{A}\tilde{B}} = \widetilde{AB}$ for $A, B \in I(R)$, so that $\mathcal{D}(R)$, together with the operation $+$ defined by $\text{div}_R(A) + \text{div}_R(B) = \text{div}_R(AB)$ is a commutative semigroup with identity $0 = \text{div}_R(R)$. Furthermore, $\mathcal{D}(R)$ is a group iff R is completely integrally closed. If we define $\leq$ on $\mathcal{D}(R)$ by $\text{div}_R(A) \leq \text{div}_R(B)$ iff $A < B$, then $\mathcal{D}(R)$ is a lattice-ordered semigroup.

2. An application of $\mathcal{D}(R)$

It is shown in [1] that the proper prime ideals in a Dedekind domain are divisorial. We now apply the general theory of $\mathcal{D}(R)$ to show that this property characterizes Dedekind domains in the class of one-dimensional Prüfer domains.

Theorem 2. 1. *Let R be a Prüfer domain in which proper prime ideals are maximal. Then R is a Dedekind domain iff every minimal prime ideal of R is divisorial.*

PROOF. ($\Leftarrow$) Since R is a one-dimensional Prüfer domain, R_M is a rank one valuation ring for each maximal ideal M of R . By [5, page 94], $R = \bigcap R_M$, where M runs over the collection of maximal ideals of R . Since R_M is completely integrally closed

for each maximal ideal M of R , it follows that R is completely integrally closed, so that $\mathcal{D}(R)$ is a group (see [1]). Let P be any non-zero maximal (hence minimal) prime of R . Then $P(R:P) \subseteq R$. If $P(R:P) \neq R$, then $P(R:P) \subseteq M$ for some maximal ideal M of R . Then $P^2 \subseteq P(R:P) \subseteq M$; i.e., $P^2 \subseteq M$. Thus $P \subseteq M$, and so $P=M$ since both M and P are minimal (maximal). So we have $P(R:P) \subseteq P$ if $P(R:P) \neq R$. But then in $\mathcal{D}(R)$ we have $0 = \text{div}_R(P(R:P)) \cong \text{div}_R(P) > 0$, a contradiction ($\text{div}_R(P) > 0$ since $P \subseteq R$, and $\tilde{P} = P < R$). So we must have $P(R:P) = R$; i.e., P is invertible and hence finitely generated. Thus R is a ring in which every prime ideal is finitely generated and so by Cohen's theorem, [6, page 8] (see also [7]), R is Noetherian. We already have that R is integrally closed and that prime ideals are maximal. It follows that R is a Dedekind domain.

($\Rightarrow$) See [1].

In [4], Gilmer defines an almost-Dedekind domain (AD -domain) to be a domain R such that R_M is a Dedekind domain for each maximal ideal M of R . It follows that if R is an almost-Dedekind domain then R is a Prüfer domain in which prime ideals are maximal. We can now state the following corollary.

Corollary 2.2. Let R be an almost-Dedekind domain. Then R is Dedekind iff proper prime ideals of R are divisorial.

3. families of valuations and the construction of $\mathcal{A}(R)$

In this section we obtain a characterization of an AD -domain R in terms of a family F of valuations on K . The family is used to construct a partially ordered semigroup $\mathcal{A}(R)$ of fractionary ideal classes and the relation between $\mathcal{A}(R)$ and $\mathcal{D}(R)$ is described.

Definition 3.1. Let v be a rank one, discrete valuation on K which is non-negative on R . For any fractionary ideal A of R , put $v(A) = \inf_{a \in A} v(a)$.

Theorem 3.2. R is AD iff there is a family F of valuations on K such that.

- (i) Each $v \in F$ has rank one and is discrete.
- (ii) $R = \bigcap_{v \in F} R_v$.
- (iii) $R_v = R_{P(v)}$, where $P(v)$ denotes the center of v on R for each $v \in F$.
- (iv) R is the only ideal A of R such that $v(A) = 0$ for all $v \in F$.

PROOF. Suppose F is a family of valuations on K which satisfies (i), (ii), (iii), (iv) above, and let P be any proper prime ideal of R . By (ii), $v(P) \cong 0$ for all $v \in F$. By (iv) there is $v \in F$ such that $v(P) > 0$. Then $(0) < P \subseteq P(v) < R$, where $P(v)$ denotes the center of v on R . Since $P(v)$ is the center of a rank one valuation, $P(v)$ is a minimal prime in R . Thus $P = P(v)$ and $R_P = R_{P(v)} = R_v$ is a rank one, discrete valuation ring. It follows that R is AD .

Suppose R is AD . Let F be the family of valuations on K induced by the family of proper primes of R . It is easy to see that F satisfies (i), (ii), (iii). To see that F satisfies (iv), let A be any proper ideal of R . Then $A \subseteq M$ for some maximal (and hence minimal prime) ideal M of R . If v denotes the valuation on K induced by

M , then $v \in F$. Furthermore, $v(A) \cong v(M) > 0$. Thus R must be the only ideal A of R such that $v(A) = 0$ for all $v \in F$, and (iv) holds.

We note that condition (iv) insures that proper prime ideals are maximal and that valuation rings R_v are identical with the quotient rings R_P where P runs over all minimal primes of R . We call F the family of essential valuations of R . In the remainder of this section, R denotes an AD -domain, F the family of essential valuations of R .

We state the following lemmas:

Lemma 3.3. *If $A, B \in I(R)$, $v \in F$, then $v(AB) = v(A) + v(B)$.*

Lemma 3.4. *For any principal fractionary ideal Rx and any $v \in F$, $v(Rx) = v(x)$.*

Now, for $A, B \in I(R)$, define $A \sim B$ iff $v(A) = v(B)$ for all $v \in F$. Then $\sim$ is an equivalence relation on $I(R)$. For $A \in I(R)$ we let $[A]$ denote the equivalence class of A with respect to $\sim$ and we let $\mathcal{A}(R)$ denote the set of all such equivalence classes. We define $+$ on $\mathcal{A}(R)$ by $[A] + [B] = [AB]$, and we define $\leq$ on $\mathcal{A}(R)$ by $[A] \leq [B]$ iff $v(A) \leq v(B)$ for each $v \in F$. With these definitions $\mathcal{A}(R)$ is a commutative, partially ordered semigroup with identity $0 = [R]$.

Lemma 3.5. *Let $A \in I(R)$. Then, considering $[A]$ and $\text{div}_R(A)$ as subsets of $I(R)$, $[A] \subseteq \text{div}_R(A)$.*

PROOF. Let $B \in [A]$. Then $v(B) = v(A)$ for each $v \in F$. If $A \subseteq Rx$, then $v(B) = v(A) \cong v(Rx) = v(x)$ for each $v \in F$. Thus if $b \in B$, $v(b) - v(x) \cong 0$ for all $v \in F$; i.e., $v\left(\frac{b}{x}\right) \cong 0$ for all $v \in F$, so $\frac{b}{x} \in \bigcap_{v \in F} R_v = R$, and hence $B \subseteq Rx$. Similarly, if $B \subseteq Ry$ then $A \subseteq Ry$. Then $\tilde{A} = \tilde{B}$ and $\text{div}_R(A) = \text{div}_R(B)$. But then $B \in \text{div}_R(A)$.

Proposition 3.6. *The map $g: \mathcal{A}(R) \rightarrow \mathcal{D}(R)$, defined by $g([A]) = \text{div}_R(A)$, is an order-preserving homomorphism of the partially ordered semigroup $\mathcal{A}(R)$ onto the lattice-ordered group $\mathcal{D}(R)$.*

PROOF. Lemma 3.5 shows that g is well-defined and onto. It is easy to check that g is a homomorphism. To see that g preserves order, recall that $\text{div}_R(A) \leq \text{div}_R(B)$ iff $A \leq B$. We shall show that if $[A] \leq [B]$, then $A \leq B$. Thus suppose $[A] \leq [B]$ and let $A \subseteq Rx$. Then $v(B) \cong v(A) \cong v(x)$ for each $v \in F$ since $[A] \leq [B]$. As in the proof of 3.5 we have $B \subseteq Rx$ so that $A \leq B$.

We can now prove the following theorem.

Theorem 3.7. *Let R be an AD -domain with family F of essential valuations. The following statements are equivalent.*

- (1) *Every proper prime ideal of R is divisorial.*
- (2) *Every fractionary ideal of R is divisorial.*
- (3) *R is a Dedekind domain.*
- (4) *$\mathcal{A}(R)$ is a group.*
- (5) *The map g of $\mathcal{A}(R)$ onto $\mathcal{D}(R)$ is an isomorphism.*

PROOF. (1) $\Leftrightarrow$ (3). This is corollary 2.2.

(2) $\Rightarrow$ (5) If every fractionary ideal is divisorial, then for $A \in I(R)$ we have $\text{div}_R(A) = \{A\}$. But $A \in [A] \subseteq \text{div}_R(A) = \{A\}$. It follows that, considered as sets, $[A] = \text{div}_R(A)$ so that g is 1—1, hence an isomorphism.

- (5) $\Rightarrow$ (4) Clear since $\mathcal{D}(R)$ is a group.
 (4) $\Rightarrow$ (3) Suppose $\mathcal{A}(R)$ is a group. Then if $[A] \in \mathcal{A}(R)$, there is $[B] \in \mathcal{A}(R)$ such that $[A] + [B] = 0$; i.e., $[AB] = [R]$. Then $v(AB) = 0$ for all $v \in F$ and so by theorem 3.2, $AB = R$. Thus A is invertible and hence has a finite basis so that R is Noetherian and therefore Dedekind.
 (3) $\Rightarrow$ (2) This is found in [1], page 23.

4. Some further examples of AD-domains

Let R be an AD-domain with quotient field K and family F of essential valuations. Let L be a subfield of K and put $A = R \cap L$. We may suppose that L is the quotient field of A , for if T is the quotient field of A then $A \subseteq T \subseteq L$, and so $A \subseteq R \cap T \subseteq R \cap L = A$.

Proposition 4.1. *If R is integral over A , then A is almost-Dedekind.*

PROOF. For $v \in F$, let v' denote the restriction of v to L . Let $F' = \{v' | v \in F \text{ and } v \text{ is nontrivial on } L\}$. Thus if $v' \in F'$, then $v'(x) \neq 0$ for some non-zero element $x \in L$. It is clear that each $v' \in F'$ is a rank one discrete valuation since each $v \in F$ is a rank one discrete valuation. Since $R = \bigcap_{v \in F} R_v$ we have $A = R \cap L = \left(\bigcap_{v \in F} R_v \right) \cap L = \bigcap_{v \in F} (R_v \cap L) = \bigcap_{v' \in F'} (R_v \cap L)$, for if $w \in F$ is trivial on L , then $L \subseteq R_w$. For $v' \in F'$, let $A_{v'} = \{x \in L | v'(x) \geq 0\}$, and let $Q(v') = \{y \in A | v'(y) > 0\}$. Then $Q(v') = A \cap P(v)$, so $Q(v')$ is a non-zero minimal prime of A since $P(v)$ is a non-zero minimal prime of R and R is integral over A . We have $A_{Q(v')} = L \cap R_{P(v)} = L \cap R_v = A_{v'}$ for each $v' \in F'$. For if $\frac{x}{y} \in A_{Q(v')}$, then $v' \left(\frac{x}{y} \right) \geq 0$; i.e., $\frac{x}{y} \in L \cap R_v = L \cap R_{P(v)} = A_{v'}$. Thus $A_{Q(v')} \subseteq A_{v'}$. On the other hand, let $\frac{x}{y} \in A_{v'}$, $x, y \in A$. Then $v' \left(\frac{x}{y} \right) \geq 0$. Since v' is a non-trivial, rank one discrete valuation on L , there is an irreducible element $\pi \in A_{v'}$, such that $\frac{x}{y} = \frac{\pi^n u}{w}$, where $n \geq 0$ is an integer, $u, w \in A$ are such that $v(u) = 0 = v(w)$ and $\pi \in A$ without loss of generality. Then $w \in Q(v')$, and so $\frac{x}{y} \in A_{Q(v')}$; i.e., $A_{v'} = A_{Q(v')}$. Now let B be an ideal of A such that $v'(B) = 0$ for all $v' \in F'$, where $v'(B) = \inf_{b \in B} v'(b)$, for each $v' \in F'$. Then $v(BR) = 0$ for all $v \in F$. For if $v \in F$ is trivial on L , then $v(BR) = 0$ clearly. If $v \in F$ is such that $v' \in F'$, then $v'(B) = v(BR) = 0$. Thus BR is an ideal of R such that $v(BR) = 0$ for all $v \in F$, and so $BR = R$. In this case, $B = A$. For if $B < A$, then $B \subseteq M'$ for some maximal ideal M' of A . Since R is integral over A , there is a maximal ideal M of R such that $M \cap A = M'$. Then $B \subseteq M$, and so $BR \subseteq M < R$, a contradiction. So we must have $B = A$. Thus F' is a family of valuations on L satisfying the conditions of theorem 3.2.

Proposition 4.2. *Let R be an AD-domain and let x be an indeterminate. Let S denote the multiplicatively closed set of $R[x]$ consisting of all monic polynomials. Then $(R[x])_S$ is AD.*

PROOF. Denote $(R[x])_S$ by R^1 . Let $\mathcal{P}$ be a prime ideal in R^1 , $(0) < \mathcal{P} < R^1$, and consider $R^1_{\mathcal{P}}$.

Case 1. $\mathcal{P} \cap R = (0)$. Then $K[x] \subseteq R^1_{\mathcal{P}}$ and $K[x]$ is Dedekind. Hence $R^1_{\mathcal{P}}$ is Dedekind.

Case 2. $\mathcal{P} \cap R = P \neq (0)$. Then P is a prime ideal in R , and $\mathcal{P} \cap R[x] = PR[x]$. For if not, then $\mathcal{P} \cap S \neq \emptyset$, and $\mathcal{P} = R^1$, a contradiction.

Let $M = R^1 - \mathcal{P}$. Then $M_1 = R[x] - PR[x] \subseteq M$. Now $R[x]_P = R_P[x] \subseteq R^1_{\mathcal{P}}$. Let S' denote the set of monic polynomials in $R_P[x]$. Then $(R_P[x])_{S'}$ is Dedekind since R_P is Dedekind by Lemma 2.1 of [2]. We have $(R_P[x])_{S'} \subseteq R^1_{\mathcal{P}}$. For, let $z \in (R_P[x])_{S'} = (R[x]_P)_{S'}$. Then

$$z = \frac{f(x)}{m} \left/ x^n + \frac{a_{n-1}}{m_{n-1}} x^{n-1} + \dots + \frac{a_0}{m_0} \right., \text{ where } f(x) \in R[x], m, m_0, \dots, m_{n-1} \in R - P, \\ a_0, \dots, a_{n-1} \in R.$$

Then $z = \frac{f(x)}{m} \left/ [(m'x^n + a'_{n-1}x^{n-1} + \dots + a'_0)/m'] \right.$, where $m' = m_0m_1 \dots m_{n-1}$. Then $z = m'f(x)/(mm'x^n + \dots + ma'_0)$ and $g(x) = mm'x^n + \dots + ma'_0$ does not belong to $\mathcal{P}$ since $mm' \in R - P$, and so $g(x) \in R[x] - PR[x]$. Thus, $z \in R^1_{\mathcal{P}}$. Since $(R_P[x])_{S'} \subseteq R^1_{\mathcal{P}}$ and $(R_P[x])_{S'}$ is Dedekind, $R^1_{\mathcal{P}}$ is also Dedekind.

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