

# On Riesz summability of a series of Bessel functions

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## 1. Introduction

Let  $c_v(\alpha, \beta) \equiv J_v(\alpha)Y_v(\beta) - Y_v(\alpha)J_v(\beta)$ , where  $J_v(z)$  and  $Y_v(z)$  denote the Bessel functions of first and second kind of order  $v > 0$ .

Let  $\gamma_m$  denote the  $m$ th positive root of the equation

$$c_v(az, bz) = 0.$$

An arbitrary function  $f(x)$  defined for  $0 < a < x < b$  can be expanded in a series of the form

$$(1.1) \quad f(x) \sim \sum_{m=1}^{\infty} b_m c_v(\gamma_m x, \gamma_m b),$$

where

$$(1.2) \quad b_m = \frac{\pi^2 \gamma_m^2}{2} \frac{J_v^2(\gamma_m a)}{\{J_v^2(\gamma_m a) - J_v^2(\gamma_m b)\}} \int_a^b t f(t) c_v(\gamma_m t, \gamma_m b) dt.$$

E. C. TITCHMARSH ([3]) studied the convergence of the series (1.1) and gave the following theorem:

**Theorem A.** *If  $f(t)$  be Lebesgue integrable in the range  $(a, b)$  and has bounded variation in the neighbourhood of the point  $t = x$ , then the series (1.1) converges to the sum  $\frac{1}{2}\{f(x+0) + f(x-0)\}$ .*

The purpose of this paper is to study the Riesz summability of the series (1.1).

G. N. WATSON ([5], pp. 606—607) has studied the Riesz summability of the ordinary Fourier-Bessel series

$$(1.3) \quad f(x) \sim \sum_{m=1}^{\infty} a_m J_v(j_m x),$$

where  $j_1 < j_2 < \dots < j_m < \dots$  are the positive zeros of the function  $J_v(z)$  and

$$(1.4) \quad a_m = \frac{2}{J_{v+1}^2(j_m)} \int_0^1 t f(t) J_v(j_m t) dt.$$

He has taken the Riesz mean as the sum

$$(1.5) \quad \lim_{n \rightarrow \infty} \sum_{m=1}^n \left(1 - \frac{j_m}{A_n}\right) a_m J_v(j_m x),$$

where  $A_n$  stands for  $(n + \frac{1}{2}v + \frac{1}{4})\pi$  and  $j_n < A_n < j_{n+1}$ . The asymptotic value of  $j_m$  is known to be ([5], p. 618) given as

$$(1.6) \quad j_m = \left(m + \frac{1}{2}v - \frac{1}{4}\right)\pi + O(m^{-1}).$$

Analogous to above, we define the Riesz sum for the series (1.1) by

$$(1.7) \quad R_n(x, f) \sim \lim_{n \rightarrow \infty} \sum_{m=1}^n \left(1 - \frac{\gamma_m}{\beta_n}\right) b_m c_v(\gamma_m x, \gamma_m b),$$

where  $\beta_n$  lies between  $\gamma_n$  and  $\gamma_{n+1}$  and\*)

$$(1.8) \quad \gamma_m = \frac{m\pi}{b-a} + \frac{4(v^2-1)(b-a)}{8m\pi ab} + O(m^{-3}).$$

The series (1.1) is said to be summable-(R) if the limit in (1.7) exists. In § 3 we shall prove the following theorem.

**Theorem:** If  $v > 0$  and

$$(1.9) \quad \int_a^b t^{1/2} |f(t)| dt < \infty,$$

and if the limits  $f(x \pm 0)$  exist, then the series (1.1) is summable-(R) at all points of the open interval  $(a, b)$  to the sum

$$\frac{1}{2} \{f(x+0) + f(x-0)\}.$$

2. The following lemmas will be needed during the course of the proof of the theorem:

**Lemma 1.** On the rectangle whose vertices are  $\pm \beta i$ ,  $\beta_n \pm \beta i$  in the  $\omega$  plane, where  $\beta$  will be made to tend to infinity, we have

$$(2.1) \quad \left| \frac{\omega c_v(\omega x, a\omega) c_v(\omega t, b\omega)}{c_v(a\omega, b\omega)} \right| = O\left(\frac{e^{-v(t-x)}}{\sqrt{xt}}\right)$$

for  $t > x$  and  $\omega = u + iv$ .

PROOF. Titchmarsh ([4], p. 73) has obtained the relation

$$(2.2) \quad J_v(bs) Y_v(xs) - Y_v(bs) J_v(xs) = -\frac{2 \sin \{s(b-x)\}}{\sqrt{bx} \pi s} + O\left(\frac{e^{|t|(b-x)}}{|s|^2}\right),$$

where  $s = \sigma + it$  and  $0 < x < b$ .

\*) See NAYLOR [2].

Using the above relation we easily have

$$(2.3) \quad c_v(x\omega, a\omega) = -\frac{2 \sin \{\omega(x-a)\}}{\sqrt{ax} \pi \omega} + O \left\{ \frac{e^{|v|(x-a)}}{|\omega|^2} \right\}, \quad \text{for } 0 < a < x;$$

$$(2.4) \quad c_v(\omega t, \omega b) = \frac{2 \sin \{\omega(b-t)\}}{\sqrt{bt} \pi \omega} + O \left\{ \frac{e^{|v|(b-t)}}{|\omega|^2} \right\}, \quad \text{for } 0 < t < b;$$

and

$$(2.5) \quad c_v(a\omega, b\omega) = \frac{2 \sin \{\omega(b-a)\}}{\sqrt{ab} \pi \omega} + O \left\{ \frac{e^{|v|(b-a)}}{|\omega|^2} \right\}, \quad \text{for } 0 < a < b.$$

Also, since on the above mentioned rectangle in the  $\omega$  plane [4, pp. 13—14]

$$|\sin \{\omega(b-a)\}| > Ae^{v(b-a)},$$

we have

$$(2.6) \quad \left| \omega \frac{c_v(x\omega, a\omega) c_v(\omega t, b\omega)}{c_v(a\omega, b\omega)} \right| < |\omega| \left[ \left| \frac{\sin \{\omega(x-a)\} \sin \{\omega(b-t)\}}{\pi \sqrt{xt} \omega \sin \{\omega(b-a)\}} \right| + O \left\{ \frac{e^{|v|(x-t)}}{|\omega|^2} \right\} \right] =$$

$$= O \left( \frac{e^{-v(t-x)}}{\sqrt{xt} \pi} \right) + O \left( \frac{e^{-v(t-x)}}{|\omega|} \right) = O \left( \frac{e^{-v(t-x)}}{\sqrt{xt}} \right).$$

Thus the proof of the lemma is complete.

*Riesz sum:* The Riesz sum of the series (1.1) is given by

$$(2.7) \quad R_n(x, f) = \sum_{m=1}^n \left( 1 - \frac{\gamma_m}{\beta_n} \right) b_m c_v(\gamma_m x, \gamma_m b)$$

$$= \sum_{m=1}^n \left( 1 - \frac{\gamma_m}{\beta_n} \right) \frac{\pi^2 \gamma_m^2 J_v^2(\gamma_m a) c_v(\gamma_m x, \gamma_m b)}{2 \{ J_v^2(\gamma_m a) - J_v^2(\gamma_m b) \}} \int_a^b t f(t) c_v(\gamma_m t, \gamma_m b) dt$$

$$(2.8) \quad = \int_a^b t f(t) R_n(t, x/R) dt,$$

where

$$(2.9) \quad R_n(t, x/R) = \sum_{m=1}^n \left( 1 - \frac{\gamma_m}{\beta_n} \right) \frac{\pi^2 \gamma_m^2 J_v^2(\gamma_m a) c_v(\gamma_m x, \gamma_m b) c_v(\gamma_m t, \gamma_m b)}{2 \{ J_v^2(\gamma_m a) - J_v^2(\gamma_m b) \}}.$$

In order to study the relevant properties of the kernel function  $R_n(t, x/R)$  we observe that it is a symmetrical function in  $t$  and  $x$  and hence we first suppose that  $a < x < t < b$  and subsequently the results for  $a < t < x < b$  can be written by an interchange of  $t$  and  $x$ .

**Lemma 2.** *The following inequalities hold true\*):*

$$(2.10) \quad |R_n(t, x|R)| < \frac{2K}{\beta_n} \frac{1}{\sqrt{xt}(t-x)^2}, \quad \text{if } a < x < t < b;$$

$$(2.11) \quad |R_n(t, x|R)| < \frac{2K}{\beta_n} \frac{1}{\sqrt{xt}(x-t)^2}, \quad \text{if } a < t < x < b;$$

$$(2.12) \quad |R_n(t, x|R)| < \frac{3\beta_n K}{\sqrt{\frac{1}{2}xt}}, \quad \text{if } a < x \leq t < b \quad \text{or} \quad a < t \leq x < b.$$

PROOF. Consider the function

$$(2.13) \quad \Phi(\omega) = \pi\omega \left(1 - \frac{\omega}{\beta_n}\right) \frac{c_v(x\omega, a\omega)c_v(t\omega, b\omega)}{c_v(a\omega, b\omega)}.$$

This function has simple poles at  $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n$ . Residue at  $\omega = \gamma_m$  is given by

$$\begin{aligned} (2.14) \quad \lim_{\omega \rightarrow \gamma_m} \pi\omega \left(1 - \frac{\omega}{\beta_n}\right) (\omega - \gamma_m) \frac{c_v(x\omega, a\omega)c_v(t\omega, b\omega)}{c_v(a\omega, b\omega)} &= \\ &= \lim_{\omega \rightarrow \gamma_m} \times \\ &\times \frac{\pi\omega \left(1 - \frac{\omega}{\beta_n}\right) c_v(x\omega, a\omega)c_v(t\omega, b\omega) + (\omega - \gamma_m) \frac{d}{d\omega} \left[ \pi\omega \left(1 - \frac{\omega}{\beta_n}\right) c_v(x\omega, a\omega)c_v(t\omega, b\omega) \right]}{c'_v(a\omega, b\omega)} = \\ &= \frac{\pi\gamma_m \left(1 - \frac{\gamma_m}{\beta_n}\right) c_v(\gamma_m x, \gamma_m a)c_v(\gamma_m t, \gamma_m b)}{a \{J'_v(\gamma_m a) Y_v(\gamma_m b) - J_v(\gamma_m b) Y'_v(\gamma_m a)\} + b \{J_v(\gamma_m a) Y'_v(\gamma_m b) - J'_v(\gamma_m b) Y_v(\gamma_m a)\}} = \\ &= \frac{\pi\gamma_m \left(1 - \frac{\gamma_m}{\beta_n}\right) c_v(\gamma_m x, \gamma_m a)c_v(\gamma_m t, \gamma_m b)}{a \left\{ J'_v(\gamma_m a) \frac{Y_v(\gamma_m a)}{\eta} - \frac{J_v(\gamma_m a)}{\eta} Y'_v(\gamma_m a) \right\} + b \{ \eta J_v(\gamma_m b) Y'_v(\gamma_m b) - J'_v(\gamma_m b) \eta Y_v(\gamma_m b) \}} \end{aligned}$$

where

$$(2.15) \quad \eta = \frac{J_v(\gamma_m a)}{J_v(\gamma_m b)} = \frac{Y_v(\gamma_m a)}{Y_v(\gamma_m b)},$$

$$\frac{\pi\gamma_m \left(1 - \frac{\gamma_m}{\beta_n}\right) c_v(\gamma_m x, \gamma_m a)c_v(\gamma_m t, \gamma_m b)}{\frac{1}{\eta} \left( -\frac{2}{\pi\gamma_m} \right) + \eta \left( \frac{2}{\pi\gamma_m} \right)} = - \frac{\pi^2 \gamma_m^2 \left(1 - \frac{\gamma_m}{\beta_n}\right) c_v(\gamma_m x, \gamma_m a)c_v(\gamma_m t, \gamma_m b)}{2 \left( \eta - \frac{1}{\eta} \right)},$$

\*)  $K$  will denote a constant not necessarily the same on each occasion.

since

$$(2.16) \quad J_\nu(x) Y'_\nu(x) - Y_\nu(x) J'_\nu(x) = \frac{2}{\pi x}.$$

Using (2.15) again and after a little simplification we find that the residue at  $\omega = \gamma_m$  is

$$(2.17) \quad \frac{\pi^2 \gamma_m^2 \left(1 - \frac{\gamma_m}{\beta_n}\right) J_\nu(\gamma_m a) J_\nu(\gamma_m b) [J_\nu(\gamma_m x) Y_\nu(\gamma_m a) - J_\nu(\gamma_m a) Y_\nu(\gamma_m x)] c_\nu(\gamma_m t, \gamma_m b)}{2 \{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)\}} =$$

$$= \frac{\pi^2 \gamma_m^2 \left(1 - \frac{\gamma_m}{\beta_n}\right) J_\nu^2(\gamma_m a) c_\nu(\gamma_m x, \gamma_m b) c_\nu(\gamma_m t, \gamma_m b)}{2 \{J_\nu^2(\gamma_m a) - J_\nu^2(\gamma_m b)\}}$$

Therefore  $R_n(t, x|R)$  is the sum of the residues of the function  $\Phi(\omega)$  at  $\gamma_1, \gamma_2, \dots, \gamma_n$ .

*Proof of inequality (2.10).* We take the contour of integration as the rectangle with the vertices at  $\pm \beta i, \beta_n \pm \beta i$  in the  $\omega$  plane, where  $\beta$  will be made to tend to  $\infty$ .

Hence

$$(2.18) \quad R_n(t, x|R) = \frac{1}{2\pi i} \left[ \int_{\beta_n - i\infty}^{\beta_n + i\infty} - \int_{-i\infty}^{i\infty} \right] \frac{\pi \omega \left(1 - \frac{\omega}{\beta_n}\right) c_\nu(\omega x, \omega a) c_\nu(\omega t, \omega b)}{c_\nu(a\omega, b\omega)} d\omega,$$

as it can be easily shown in view of Lemma 1 that the other parts of the integral tend to zero.

Now

$$(2.19) \quad R_n(t, x|R) = \frac{1}{2i} \int_{\beta_n - i\infty}^{\beta_n + i\infty} \omega \left(1 - \frac{\omega}{\beta_n}\right) \frac{c_\nu(\omega x, a\omega) c_\nu(t\omega, b\omega)}{c_\nu(a\omega, b\omega)} d\omega +$$

$$+ \frac{1}{2i\beta_n} \int_{-i\infty}^{i\infty} \frac{\omega^2 c_\nu(\omega x, a\omega) c_\nu(t\omega, b\omega)}{c_\nu(a\omega, b\omega)} d\omega,$$

since  $\frac{\omega c_\nu(\omega x, a\omega) c_\nu(t\omega, b\omega)}{c_\nu(a\omega, b\omega)}$  is an odd function of  $\omega$ .

Using Lemma 1, and replacing  $\omega$  by  $\beta_n \pm iv$  and  $\pm iv$  in the first and second integrals respectively, we obtain

$$(2.20) \quad |R_n(t, x|R)| < \frac{2k}{\sqrt{xt}} \frac{1}{\beta_n} \int_0^\infty v e^{-(t-x)v} dv = \frac{2k}{\beta_n} \frac{1}{\sqrt{xt}} \frac{1}{(t-x)^2},$$

which completes the proof of inequality (2.10). The inequality (2.11) follows by an interchange of  $x$  and  $t$ .

*Proof of inequality (2.12).* In this case we take the contour as the rectangle whose vertices are  $\pm i\beta_n$  and  $\beta_n \pm i\beta_n$  in the  $\omega$ -plane.

Using the result of Lemma 1, we find that

$$(2.21) \quad \left| \left( 1 - \frac{\omega}{\beta_n} \right) \frac{\omega c_v(x\omega, a\omega) c_v(t\omega, b\omega)}{c_v(a\omega, b\omega)} \right| \leq \frac{k}{\sqrt{xt}} \sqrt{2},$$

when  $a < x \leq t < b$  and the same inequality will hold for  $a < t \leq x < b$  as can be seen by interchanging  $x$  and  $t$ .

Consequently

$$(2.22) \quad |R_n(t, x|R)| < \frac{3k\beta_n}{\sqrt{\frac{1}{2}xt}}.$$

**Lemma 3.**

$$\lim_{n \rightarrow \infty} \int_a^b t^{v+1} R_n(t, x|R) dt = x^v \quad (a < x < b).$$

PROOF. From (2.9) we have

$$(2.23) \quad \int_a^b t^{v+1} R_n(t, x|R) dt = \sum_{m=1}^n \left( 1 - \frac{\gamma_m}{\beta_n} \right) \frac{\pi^2 \gamma_m^2 J_v^2(\gamma_m a) c_v(\gamma_m x, \gamma_m b)}{2[J_v^2(\gamma_m a) - J_v^2(\gamma_m b)]} \int_a^b t^{v+1} c_v(\gamma_m t, \gamma_m b) dt.$$

Using (2.9) again it can be seen easily that

$$\begin{aligned} &= \sum_{m=1}^n \left( 1 - \frac{\gamma_m}{\beta_n} \right) \frac{\pi^2 \gamma_m^2 J_v^2(\gamma_m a) c_v(\gamma_m x, \gamma_m b)}{2[J_v^2(\gamma_m a) - J_v^2(\gamma_m b)]} \left[ \frac{b^{v+1}}{\gamma_m} \left( \frac{2}{\pi \gamma_m b} \right) - \frac{a^{v+1}}{\eta \gamma_m} \left( \frac{2}{\pi \gamma_m a} \right) \right] = \\ &= \sum_{m=1}^n \frac{\pi \left( 1 - \frac{\gamma_m}{\beta_n} \right) \left\{ b^v J_v^2(\gamma_m a) - \frac{J_v(\gamma_m b)}{J_v(\gamma_m a)} J_v^2(\gamma_m a) a^v \right\} c_v(\gamma_m x, \gamma_m b)}{J_v^2(\gamma_m a) - J_v^2(\gamma_m b)} = \\ &= \sum_{m=1}^n \pi \left( 1 - \frac{\gamma_m}{\beta_n} \right) J_v(\gamma_m a) \frac{b^v J_v(\gamma_m a) - a^v J_v(\gamma_m b)}{J_v^2(\gamma_m a) - J_v^2(\gamma_m b)} c_v(\gamma_m x, \gamma_m b), \end{aligned}$$

and this is evidently the sum of the residues of the function

$$\left( 1 - \frac{\omega}{\beta_n} \right) \frac{2b^v c_v(\omega x, a\omega) - 2a^v c_v(x\omega, b\omega)}{\omega c_v(a\omega, b\omega)}$$

at  $\gamma_1, \gamma_2, \dots, \gamma_n$ . Therefore transforming the sum into a contour integral as in Lemma 2, we have

$$(2.24) \quad \int_a^b t^{v+1} R_n(t, x|R) dt = \frac{1}{2\pi i} \int_C \left( 1 - \frac{\omega}{\beta_n} \right) \frac{2b^v c_v(x\omega, a\omega) - 2a^v c_v(x\omega, b\omega)}{\omega c_v(a\omega, b\omega)} d\omega$$

where  $c'$  is the contour indented at the origin in the usual manner by a semi-circle to the right of the imaginary axis.

In view of the Lemma 1, we obtain that

$$(2.25) \quad \int_a^b t^{v+1} R_n(t, x|R) dt = \frac{1}{2\pi i} \left[ - \int_{-\infty i}^{\infty i} \left( 1 - \frac{\omega}{\beta_n} \right) \frac{2b^v c_v(x\omega, a\omega) - 2a^v c_v(\omega x, b\omega)}{\omega c_v(a\omega, b\omega)} d\omega + \right. \\ \left. + \int_{\beta_n - i\infty}^{\beta_n + i\infty} \left( 1 - \frac{\omega}{\beta_n} \right) \frac{2b^v c_v(x\omega, a\omega) - 2a^v c_v(\omega x, b\omega)}{\omega c_v(a\omega, b\omega)} d\omega \right].$$

Since the function  $\frac{2b^v c_v(\omega x, a\omega) - 2a^v c_v(\omega x, b\omega)}{\omega c_v(a\omega, b\omega)}$  is an odd function of  $\omega$ , the value of the integral on the right of (2.25) reduces to  $\pi i$  times the residue of the integrand at the origin as  $n \rightarrow \infty$ .

Now, residue of the integrand at the simple pole  $w=0$  is

$$\lim_{\omega \rightarrow 0} \frac{2b^v c_v(\omega x, a\omega) - 2a^v c_v(\omega x, b\omega)}{c_v(a\omega, b\omega)},$$

which gives after some simplification

$$(2.26) \quad \lim_{\omega \rightarrow 0} \frac{2b^v \frac{J_v(a\omega)}{J_v(b\omega)} \frac{Y_v(a\omega)}{Y_v(b\omega)} \left\{ \frac{J_v(x\omega)}{J_v(a\omega)} - \frac{Y_v(x\omega)}{Y_v(a\omega)} \right\} - 2a^v \left\{ \frac{J_v(x\omega)}{J_v(b\omega)} - \frac{Y_v(x\omega)}{Y_v(b\omega)} \right\}}{\frac{J_v(a\omega)}{J_v(b\omega)} - \frac{Y_v(a\omega)}{Y_v(b\omega)}}.$$

Using now the asymptotic values ([1], p. 135)

$$J_v(xt) \sim \frac{\left( \frac{xt}{2} \right)^v}{\sqrt{v+1}}, \quad v \geq 0;$$

and

$$Y_v(xt) \sim -\frac{2^v \sqrt{v}}{\pi x^v}, \quad v > 0;$$

as  $x \rightarrow 0$  and  $t$  fixed, we obtain

$$(2.27) \quad \lim_{\omega \rightarrow 0} \frac{J_v(a\omega)}{J_v(b\omega)} = \frac{a^v}{b^v},$$

and

$$(2.28) \quad \lim_{\omega \rightarrow 0} \frac{Y_v(a\omega)}{Y_v(b\omega)} = \frac{b^v}{a^v}$$

Consequently (2. 26) becomes

$$\frac{2b^v \frac{a^v}{b^v} \frac{b^v}{a^v} \left( \frac{x^v}{a^v} - \frac{a^v}{x^v} \right) - 2a^v \left( \frac{x^v}{b^v} - \frac{b^v}{x^v} \right)}{\frac{a^v}{b^v} - \frac{b^v}{a^v}}$$

which ultimately reduces to the value  $-2x^v$ .

Coming back to (2. 25) we find that

$$\int_a^b t^{v+1} R_n(t, x|R) dt = x^v + \frac{1}{\pi i} \int_{\beta_n - i\infty}^{\beta_n + i\infty} \left( 1 - \frac{\omega}{\beta_n} \right) \frac{b^v c_v(\omega x, a\omega) - a^v c_v(\omega x, b\omega)}{\omega c_v(a\omega, b\omega)} d\omega +$$

(2. 29)

$$+ \frac{1}{\beta_n \pi i} \int_{-i\infty}^{+i\infty} \frac{b_v c_v(x\omega, a\omega) - a^v c_v(x\omega, b\omega)}{c_v(a\omega, b\omega)} d\omega = x^v + L_1,$$

say.

Now by Lemma 1

$$\begin{aligned} |L_1| &< \frac{2kb^{v+1/2}}{\beta_n^2 \sqrt{x}} \int_0^\infty v e^{-(b-x)v} dv + \frac{2ka^{v+1/2}}{\beta_n^2 \sqrt{x}} \int_0^\infty v e^{-(x-a)v} dv + \\ (2. 30) \quad &= \frac{2kb^{v+1/2}}{\beta_n \sqrt{x}} \int_0^\infty e^{-(b-x)v} dv + \frac{2ka^{v+1/2}}{\beta_n \sqrt{x}} \int_0^\infty e^{-(x-a)v} dv = \\ &= \frac{2kb^{v+1/2}}{\beta_n^2 \sqrt{x} (b-x)} + \frac{2ka^{v+1/2}}{\beta_n^2 \sqrt{x} (x-a)} + \frac{2kb^{v+1/2}}{\beta_n \sqrt{x} (b+x)} + \frac{2ka^{v+1/2}}{\beta_n \sqrt{x} (x-a)}. \end{aligned}$$

From (2. 29) and (2. 30) we get the result of the lemma.

**Lemma 4.** For any  $x \in (a, b)$

$$\lim_{n \rightarrow \infty} \int_a^x t^{v+1} R_n(t, x|R) dt = \frac{1}{2} x^v$$

$$\lim_{n \rightarrow \infty} \int_x^b t^{v+1} R_n(t, x|R) dt = \frac{1}{2} x^v$$

PROOF. TITCHMARSH ([3]) has shown that

$$\lim_{n \rightarrow \infty} \int_a^x t^{v+1} R_n(t, x) dt = \frac{1}{2} x^v \quad (a < x < b),$$



where  $R_n(t, x)$  denotes the convergence kernel of the series (1.1). The result of the lemma follows from the fact that convergence implies Riesz summability.

3. PROOF OF THE THEOREM. In order to prove the theorem it is sufficient to show that

$$(3.1) \quad S_n(x|R) \equiv \sum_{m=1}^n \left(1 - \frac{\gamma_m}{\beta_n}\right) b_m c_v(\gamma_m x, \gamma_m b) - \\ - \int_a^x t^{v+1} R_n(t, x|R) x^{-v} f(x-0) dt - \int_x^b t^{v+1} R_n(t, x|R) x^{-v} f(x+0) dt = o(1).$$

Now

$$(3.2) \quad S_n(x|R) = \int_a^b t f(t) R_n(t, x|R) dt - \\ - \int_a^x t^{v+1} R_n(t, x|R) x^{-v} f(x-0) dt - \int_x^b t^{v+1} R_n(t, x|R) x^{-v} f(x+0) dt = \\ = \int_a^x t^{v+1} \{t^{-v} f(t) - x^{-v} f(x-0)\} R_n(t, x|R) dt + \\ + \int_x^b t^{v+1} \{t^{-v} f(t) - x^{-v} f(x+0)\} R_n(t, x|R) dt = I_1 + I_2,$$

say, where

$$(3.3) \quad I_1 = \int_a^{x-\delta} + \int_{x-\delta}^{x-\frac{1}{\beta_n}} + \int_{x-\frac{1}{\beta_n}}^x \equiv P_1 + P_2 + P_3,$$

say where  $\delta > \frac{1}{\beta_n}$  for sufficiently large  $n$ ,

and

$$(3.4) \quad I_2 = \int_x^{x+\frac{1}{\beta_n}} + \int_{x+\frac{1}{\beta_n}}^{x+\delta} + \int_{x+\delta}^b \equiv Q_1 + Q_2 + Q_3,$$

say. Using (2.11) we have

$$(3.5) \quad |P_1| = \left| \int_a^{x-\delta} t^{v+1} \{t^{-v} f(t) - x^{-v} f(x-0)\} R_n(t, x|R) dt \right| \leq \\ \leq \int_a^{x-\delta} t^{v+1/2} \{t^{-v} f(t) - x^{-v} f(x-0)\} \frac{2k}{\beta_n} \frac{1}{\sqrt{x(x-t)^2}} dt < \\ < \frac{2k}{\beta_n \delta^2 \sqrt{x}} \int_a^{x-\delta} |t^{1/2} f(t) - t^{v+1/2} x^{-v} f(x-0)| dt = o(1).$$

Similarly

$$(3.6) \quad |Q_3| = o(1).$$

Now

$$(3.7)$$

$$|P_2| = \left| \int_{x-\delta}^{x-\frac{1}{\beta_n}} t^{v+1} [t^{-v} f(t) - x^{-v} f(x-0)] R_n(t, x) dt \right| \leq \int_{x-\delta}^{x-\frac{1}{\beta_n}} |t^{v+1/2}| \varepsilon \frac{2k}{\beta_n \sqrt{x(x-t)^2}} dt,$$

where  $\varepsilon > 0$  is an arbitrarily small number and  $\delta > 0$  has been chosen such that

$$|t^{-v} f(t) - x^{-v} f(x-0)| < \varepsilon, \quad x - \delta \leq t \leq x,$$

and

$$|t^{-v} f(t) - x^{-v} f(x+0)| < \varepsilon, \quad \text{if } x \leq t \leq x + \delta.$$

Therefore

$$(3.8) \quad |P_2| < \frac{2k\varepsilon}{\sqrt{x}\beta_n} \int_{x-\delta}^{x-\frac{1}{\beta_n}} \frac{t^{v+1/2}}{(x-t)^2} dt = 2kx^v \varepsilon = o(1).$$

Similarly

$$(3.9) \quad |Q_2| = o(1), \text{ if we use the inequality (2.10).}$$

Now using (2.12) we find that

$$(3.10) \quad |P_3| = \left| \int_{x-\frac{1}{\beta_n}}^x t^{v+1} [t^{-v} f(t) - x^{-v} f(x-0)] R_n(t, x) dt \right| \leq \\ \leq \int_{x-\frac{1}{\beta_n}}^x t^{v+1} \varepsilon \frac{3k\beta_n}{\sqrt{\frac{1}{2}xt}} dt < k\beta_n \varepsilon \int_{x-\frac{1}{\beta_n}}^x dt = o(1).$$

Similarly

$$(3.11) \quad |Q_1| = o(1).$$

Combining (3.5), (3.6), (3.8), (3.9), (3.10) and (3.11) we get the result.  
This completes the proof of the theorem.

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