

Ω -Groups with Composition

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Introduction

In 1956 P. J. HIGGINS investigated groups with multiple operators, so-called Ω -groups, in his famous paper [5]. The present paper is devoted to the study of sets of functions, mapping an Ω -group into itself, which form again Ω -groups. A purely axiomatic discussion is made possible by the definition of an " Ω -group with composition", more briefly called " Ω -composition group". In such a system the notions of "even" and "odd" functions can be generalized. Subsequently such Ω -composition groups are studied, in which each element is the sum of an even and an odd element. Finally, we deduce order-theoretic properties with the help of these notions. We restrict ourselves to functions of one variable.

1. Definitions and basic results

Let $\langle G, +, -, 0, \omega_1, \omega_2, \dots \rangle$ be an Ω -group with (in general non abelian) addition $+$, subtraction $-$ and zero element 0 ; $\omega_1, \omega_2, \dots$ denote the further operations (cf. [5], [6] and [7]).

An Ω -composition group is an Ω -group $\langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ with an operation " $\circ$ " of weight 2, called *composition*, fulfilling for all $g_i \in G$

$$(1) \quad (g_1 + g_2) \circ g_3 = g_1 \circ g_3 + g_2 \circ g_3$$

$$(2) \quad (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

and

$$(3) \quad \omega_i(g_1, \dots, g_{n_i}) \circ g = \omega_i(g_1 \circ g, \dots, g_{n_i} \circ g),$$

if the weight of ω_i is equal to $n_i > 0$, or

$$(4) \quad \omega_j \circ g = \omega_j,$$

if ω_j is a 0-ary operation.

An Ω -composition ring is an Ω -composition group $\langle G, +, -, 0, \circ, \cdot, \omega_3, \dots \rangle$, where $\omega_2 = \cdot$ is a binary operation, such that $\langle G, +, -, 0, \cdot \rangle$ is a ring.

The element $c \in G$ is called *constant*, if

$$(5) \quad c \circ 0 = c.$$

The set $C(G)$ of all constant elements of G is an Ω -subcomposition group. This can be easily verified. Each operation of weight 0 determines a constant element.

If there exists a right and at the same time left neutral element with respect to the composition, it will be denoted by j .

Remark. For the foundations of the theory of Ω -groups see [6]. Ω -composition groups of the form $\langle G, +, -, 0, \circ \rangle$ are called *near-rings*. A discussion can be found in [2]. Ω -composition rings of the type $\langle G, +, -, 0, \circ, \cdot \rangle$ are called *composition rings* (cf. [1] and [11]) or *TO-algebras* (see e.g. [8]).

Examples of Ω -composition groups: all sets of functions, mapping an Ω -group into itself, which are closed with respect to all operations (composition included), form Ω -composition groups, if the operations $+$, ω_i are transferred to functions in the usual manner and if composition means substitution of functions. Conversely, W. NÖBAUER proved in [10] that each Ω -composition group is isomorphic to such an Ω -composition group of functions on a suitable Ω -group.

An important example is formed by the composition ring $R[x]$ of all polynomials over a commutative ring R with multiplicative unit 1. The operations are defined as in the general case above (cf. [9]). One can prove immediately:

$$C(R[x]) = R; \quad j = x.$$

An element $g_1 \in G$ is called *even*, if for all $h \in G$ the equation

$$(6) \quad g_1 \circ (-h) = g_1 \circ h \quad \text{holds.}$$

The element $g_2 \in G$ is called *odd*, if

$$(7) \quad g_2 \circ (-h) = -g_2 \circ h, \quad \text{for all } h \in G.$$

In $R[x]$ all

$$\sum_{i=0}^n r_{2i} x^{2i}$$

are even, all

$$\sum_{i=0}^n r_{2i+1} x^{2i+1}$$

are odd.

To avoid trivial, but troublesome distinctions of several cases we postulate in addition to (1)–(3) that there is no $g \in G$, $g \neq 0$, with $g + g = 0$.

A subset S of G is called a *base for equality*, if the implication (8) holds:

$$(8) \quad g \circ s = h \circ s \quad \text{for all } s \in S \quad \text{implies} \quad g = h \quad (\text{cf. [12]}).$$

Lemma 1. *Let $C(G)$ be a base for equality. Then the element $g \in G$ is already even, resp. odd, if*

$$(9a) \quad g \circ (-c) = g \circ c \quad \text{resp.} \quad (9b) \quad g \circ (-c) = -g \circ c$$

is valid for all $c \in C(G)$.

To prove this, regard $(g \circ (-h)) \circ c = g \circ (-h \circ c)$, $g, h \in G$. $h \circ c =: c_0 \in C(G)$. In case (a) one can calculate $g \circ (-h \circ c) = g \circ (-c_0) = g \circ c_0 = (g \circ h) \circ c$, there-

fore is $(g \circ (-h)) \circ c = (g \circ h) \circ c$, which implies $g \circ (-h) = g \circ h$. In case (b) holds $(g \circ (-h)) \circ c = g \circ (-c_0) = -g \circ c_0 = (-g \circ h) \circ c$, therefore $g \circ (-h) = g \circ h$.

Lemma 2. Let $\langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ be an Ω -composition group with $j \in G$. If $g, h \in G$, g even, h odd, so is

$$(10) \quad g=h \text{ equivalent to } g=h=0.$$

This implies that 0 is the only element which is even and odd at the same time.

(10) is valid because $g=h$ implies $g = g \circ j = g \circ (-j) = h \circ (-j) = -h \circ j = -h$, and therefore $g=h=0$.

Theorem 1. a) The set $E(G)$ of all even elements of $G = \langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ forms an Ω -sub-composition group of G containing $C(G)$.

b) Let G contain j . Then $E(G) \neq G$ or $G = \{0\}$.

PROOF. a) Consider ω_r , having the weight $n_r > 0$. Then $\omega_r(g_1, \dots, g_{n_r}) \circ (-g) = \omega_r(g_1 \circ (-g), \dots, g_{n_r} \circ (-g)) = \omega_r(g_1 \circ g, \dots, g_{n_r} \circ g) = \omega_r(g_1, \dots, g_{n_r}) \circ g$, if all g_i ($1 \leq i \leq n_r$) are even. This implies $\omega_r(g_1, \dots, g_{n_r}) \in E(G)$. If ω_r has weight 0, then $\omega_r \circ (-g) = \omega_r = \omega_r \circ g$, therefore all elements determined by 0-ary operations are in $E(G)$. If g lies in $E(G)$, so does $-g$, because of $(-g) \circ (-h) = -(g \circ (-h)) = -(g \circ h) = (-g) \circ h$. This implies $0 \in E(G)$. Each $c \in C(G)$ fulfills $c \circ (-g) = -c = c \circ g$, for all $g \in G$, which shows that $C(G) \subseteq E(G)$.

b) If $j \in G$, then j is odd: $j \circ (-g) = -g = -j \circ g$. $j=0$ implies $G = \{0\}$. If j is unequal to 0, then, by Lemma 2, $j \notin E(G)$.

Theorem 2. Let $j \in G \neq \{0\}$, $\{\omega_i\}$ containing a binary operation ω_r , which is left and right distributive with respect to $+$ and with existing left and right neutral element e_r . Then

a) $e_r \in E(G)$. If ω_r is associative and if there exists a left and right inverse element $i_r(g_0)$ for $g_0 \in E(G)$, then $i_r(g_0)$ is uniquely determined and is contained in $E(G)$.

b) $C(G) \neq E(G)$.

PROOF. a) Let ω_r be an operation of the described kind. By definition we have $\omega_r(g, e_r) = \omega_r(e_r, g) = g$, for all $g \in G$. We define $h \in G$ by $h := e_r - e_r \circ (-j)$. $h \circ (-g) = e_r \circ (-g) - e_r \circ (-j) \circ (-g) = -(e_r \circ g - e_r \circ (-g)) = -h \circ g$. This shows that h is odd.

$$(11) \quad \omega_r(h, h) \circ (-g) = \omega_r(h \circ (-g), h \circ (-g)) = \omega_r(-h, -h) \circ g.$$

From $\omega_r(h, h) = \omega_r(h+0, h) = \omega_r(h, h) + \omega_r(0, h)$ one gets

$$(12) \quad \omega_r(0, h) = \omega_r(h, 0) = 0$$

and therefore

$$0 = \omega_r(h, 0) = \omega_r(h, h-h) = \omega_r(h, h) + \omega_r(h, -h).$$

Summarizing these results one gets

$$(13) \quad \omega_r(-h, -h) = -(-\omega_r(h, h)) = \omega_r(h, h).$$

Combining (11) with (13) proves that $\omega_r(h, h) \in E(G)$.

$$\begin{aligned} \omega_r(h, h) &= \omega_r(e_r - e_r \circ (-j), e_r - e_r \circ (-j)) = \\ &= \omega_r(e_r, e_r) - \omega_r(e_r, e_r \circ (-j)) - \omega_r(e_r \circ (-j), e_r) + \omega_r(-e_r, -e_r) \circ (-j) = \\ &= e_r - e_r \circ (-j) - e_r \circ (-j) + (-(-e_r)) \circ (-j) = e_r - e_r \circ (-j) = h. \end{aligned}$$

This shows that h is at the same time even and odd, by lemma 2 we get $h=0$. This means $e_r \circ j = e_r = e_r \circ (-j)$ and this implies $e_r \circ g = e_r \circ (-g)$ for all $g \in G$. Therefore $e_r \in E(G)$.

Given $g_0 \in E(G)$ with existing inverse element $i_r(g_0)$ with respect to ω_r . Let ω_r be associative. If $i'_r(g_0)$ is also an inverse element of g_0 , then from

$$\begin{aligned} i_r(g_0) &= \omega_r(i_r(g_0), e_r) = \omega_r(i_r(g_0), \omega_r(g_0, i_r(g_0))) = \\ &= \omega_r(\omega_r(i_r(g_0), g_0), i'_r(g_0)) = \omega_r(e_r, i'_r(g_0)) = i'_r(g) \end{aligned}$$

the uniqueness of the inverse element follows.

By definition, (14) holds:

$$(14) \quad \omega_r(g_0, i_r(g_0)) = e_r$$

This implies

$$\begin{aligned} \omega_r(g_0, i_r(g_0)) \circ (-g) &= \omega_r(g_0 \circ (-g), i_r(g_0) \circ (-g)) = \omega_r(g_0 \circ g, i_r(g_0) \circ (-g)) = \\ &= e_r \circ (-g) = e_r \circ g = \omega_r(g_0, i_r(g_0)) \circ g = \omega_r(g_0 \circ g, i_r(g_0) \circ g). \end{aligned}$$

Because of the proved uniqueness of the inverse element one gets $i_r(g_0) \circ g = i_r(g_0) \circ (-g)$ and therefore $i_r(g_0) \in E(G)$.

b) Just like in (13) one gets

$\omega_r(j, j) \circ (-g) = \omega_r(-g, -g) = \omega_r(-j, -j) \circ g = \omega_r(j, j) \circ g$ and therefore $\omega_r(j, j) \in E(G)$. If $\omega_r(j, j)$ is contained in $C(G)$, then $\omega_r(j, j) \circ 0 = \omega_r(0, 0) = 0$. It follows $0 = \omega_r(j, j) \circ e_r = \omega_r(e_r, e_r) = e_r$, and finally $g = \omega_r(g, e_r) = \omega_r(g, 0) = 0$ for all $g \in G$, which implies the excluded case $G = \{0\}$. Therefore $E(G) \neq C(G)$, and the theorem is completely proved.

Corollary. If G denotes an Ω -composition ring with multiplicative unit 1, then $1 \in E(G)$.

Furthermore, the following conclusions hold. Each sum of even (odd) elements is even (odd). If g is any element of G , then the composition of g with an even element is again even. In the case of abelian addition $E(G)$ is therefore a left ideal in the near-ring $\langle G, +, -, 0, \circ \rangle$.

Theorem 3. Let G be an Ω -composition ring, $j \in G$: j be no left nullifier with respect to multiplication. Then $E = E(G)$ has the same cardinal number like the set $U = U(G)$ of all odd elements of G : $E \sim U$.

PROOF. Consider the mapping $\varphi: g \rightarrow j \cdot g$ for all $g \in E$. One verifies immediately: $j \cdot g \in U$. Therefore $\varphi(E) = j \cdot E \subseteq U$. φ is injective, because $j \cdot g_1 = j \cdot g_2$ implies $g_1 = g_2$. One gets $E \sim j \cdot E \subseteq U$, and, by a similar argument, $U \sim j \cdot U \subseteq E$. This implies $U \sim j \cdot U \subseteq E \sim j \cdot E \subseteq U$, therefore $\text{card } U = \text{card } j \cdot U \subseteq \text{card } E = \text{card } j \cdot E \subseteq \text{card } U$, and from this one gets $\text{card } E = \text{card } U$.

2. Cleavable Ω -composition groups

An element g of an Ω -composition group G let be called *cleavable*, if it can be written in the form

$$(15) \quad g = \bar{g} + \underline{g} \quad \text{with} \quad \bar{g} \in E(G), \quad \underline{g} \in U(G).$$

If all elements of G are cleavable, then we say that G is *cleavable*.

Let $S(G)$ be the set of all cleavable elements of G . If the conditions of theorem 2 are fulfilled, then

$$(16) \quad C(G) \subset E(G) \subset S(G) \subset G \quad \text{holds.}$$

$E(G) \neq S(G)$ is valid, because, for example, $\omega_r(j, j) + j$ is not even, but cleavable.

The composition ring (and therefore the near-ring, too) $R[x]$ is cleavable:

$$\sum_{i=0}^n r_i x^i = \sum_{i=0}^{\left[\frac{n}{2}\right]} r_{2i} x^{2i} + \sum_{i=0}^{\left[\frac{n-1}{2}\right]} r_{2i+1} x^{2i+1}$$

We will give another example of a cleavable Ω -composition group (resp. Ω -composition ring). Let T be the near-ring (composition ring) generated by the real numbers R and the functions $x \rightarrow x$, $x \rightarrow \sin x$ and $x \rightarrow \cos x$ in the near-ring (composition ring) $C(R)$ of all continuous functions from R into R . $C(R)$ is cleavable, for each function $f(x) \in C(R)$ can be divided by

$$(17) \quad f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

into an even and an odd part.

We show the cleavability of T assuming that T is a composition ring. In the case of near-rings the proof is similar if one desits from forming products.

Let M be the subset of T containing all elements of T which can be formed from R , x , $\sin x$, $\cos x$ by a finite number of multiplications and compositions. The total number of these operations which lead to $m \in M$ we call the *step* of m .

Lemma 3. T is the set of all elements of the form

$$(18) \quad a = \sum_{\alpha \in A} a_{\alpha}$$

with $a_{\alpha} \in M$, A being a finite set of indices.

PROOF.* Let W be the set of all elements of the kind (18). $W \subseteq T$ is trivial. All the elements generating T are contained in M and therefore in W . Thus we have only to show that W is a composition ring.

If $a, b \in W$, so is $a - b$. By the distributive laws we get $a \cdot b \in W$. Take

$$a = \sum_{\alpha \in A} a_{\alpha}, \quad b = \sum_{\beta \in B} b_{\beta} \quad (a_{\alpha}, b_{\beta} \in M).$$

*) The present kind of the proof is due to Prof. W. Nöbauer, whom I wish to thank very much for his suggestions.

It follows

$$a \circ b = \left(\sum_{\alpha \in A} a_\alpha \right) \circ \left(\sum_{\beta \in B} b_\beta \right) = \sum_{\alpha \in A} a_\alpha \circ \left(\sum_{\beta \in B} b_\beta \right).$$

It is sufficient to show that for all $a_x \in M$

$$(19) \quad a_x \circ \left(\sum_{\beta \in B} b_\beta \right) \in W$$

We verify this by induction on the step of a . Is the step = 0, then $a_x = r \in R$, x , $\sin x$ or $\cos x$. For $a_x = r$, x (19) is trivial, for $a_x = \sin x$ or $\cos x$ (19) holds because of the trigonometric addition theorems. Assume now the lemma to be proved for all $a_x \in M$ with step less than n . If $a_x \in M$, then $a_x = k \cdot 1$ or $a_x = k \circ 1$ with $k, 1 \in M$. Let a_x have the step n . Then $k, 1$ have lower steps. If $a_x = k \cdot 1$, this implies

$$a_x \circ \left(\sum_{\beta \in B} b_\beta \right) = a_x \circ b = (k \cdot 1) \circ b = (k \circ b) \cdot (1 \circ b) \in W \text{ and, if } a_x = k \circ 1:$$

$$a_x \circ \left(\sum_{\beta \in B} b_\beta \right) = a_x \circ b = (k \circ 1) \circ b = k \circ (1 \circ b) \in W, \text{ because of } 1 \circ b \in W. \text{ This}$$

shows that $T = W$ and the lemma is proved.

By the help of this lemma we get now

Theorem 4. T is cleavable.

PROOF. If $a \in T$, we have by the preceding lemma:

$$a = \sum_{\alpha \in A} a_\alpha \quad (a_\alpha \in M).$$

The theorem is proved if we show that each $m \in M$ is even or odd. We do this again by induction on the step of m . If m has step 0, then again $m = r, x, \sin x$ or $\cos x$. The assertion holds, because r and $\cos x$ are even, x and $\sin x$ are odd. Now let m have step n . Again we can write m as $m = k \cdot 1$ or $m = k \circ 1$, with $k, 1 \in M$ having a lower step than n . By induction hypothesis and theorem 1 m is either even or odd. This proves the theorem.

Finally, we remark two useful statements in cleavable Ω -composition groups:

$$(20) \quad g \circ (-h) = (\bar{g} - \underline{g}) \circ h \text{ for all } g, h \in G$$

$$(21) \quad g \in U(G) \text{ implies } g \circ 0 = 0.$$

3. Applications to the theory of ordered Ω -composition groups

An Ω -composition group $\langle G, +, -, 0, \circ, \omega_2, \dots \rangle$ is called *fully* (resp. *partially*) *ordered*, if it is a fully (partially) ordered universal algebra (see [3]) and if

$$(22) \quad \langle G, +, -, 0 \rangle \text{ is a fully (partially) ordered group}$$

$$(23) \quad g, h \geq 0 \text{ implies } g \circ h \geq 0.$$

The Ω -composition group is called *fully* (*partially*) *ordered in the wider sense*, if (23) needs only to be valid if $h \notin C(G)$. (cf. [4], [11], [12]).

For Ω -composition rings one postulates furthermore that $\langle G, +, -, 0, \circ, \cdot \rangle$ is a fully (partially) ordered ring.

The symbols $|g|$ and $\text{sign } g$ are defined as usual. As an example of a composition ring which is ordered in the wider sense but not ordered take again $R[x]$ with the "lexicographic" order (cf. [11]): call $\sum_{i=0}^n r_i x^i > 0$ if and only if $r_n > 0$ in a given order in R . It can easily be shown by a counterexample that $R[x]$ cannot be ordered. The composition ring $\{\sum_{i=1}^n r_i x^i, r_i \in R\}$ is, in opposite to $R[x]$, ordered in the lexicographic order. A detailed discussion of partially and fully ordered composition rings can be found in [11] and [12].

Theorem 5. *Let G be a fully ordered, cleavable Ω -composition group with abelian addition. Let be $g \in G, h \in G$. (If G is only fully ordered in the wider sense one has to postulate $h \notin C(G)$.) Then*

$$(24) \quad |g \circ h| = |g| \circ |h|, \quad \text{if } h \geq 0 \quad \text{or} \quad g \in E \cup U.$$

$$(25) \quad |g \circ h| \leq |g| \circ |h|, \quad \text{if } h < 0 \quad \text{and} \quad \text{sign } \bar{g} = \text{sign } g.$$

$$(26) \quad |g \circ h| \geq |g| \circ |h|, \quad \text{if } h < 0 \quad \text{and} \quad \text{sign } \bar{g} \neq \text{sign } g.$$

Remark. In (25) and (26) the relations $<$ and $>$ can hold actually.

PROOF. To (24):

(i) $g \geq 0, h \geq 0$. Then $g \circ h \geq 0$ and (24) is verified.

(ii) $g < 0, h \geq 0$. Then $|g| \circ |h| = (-g) \circ h = -(g \circ h) \geq 0$, which implies (24).

(iii) $g \in E = E(G), h < 0$ (for $h \geq 0$ see (i) and (ii)). $g \circ (h) = g \circ h$. If $g \geq 0$, then $g \circ (-h) = g \circ h \geq 0$; if $g < 0$, then $g \circ h \geq 0$ and therefore $|g \circ h| = -(g \circ h) = (-g) \circ (-h) = |g| \circ |h|$.

(iv) $g \in U = U(G)$. If $g \geq 0$, then $g \circ (-h) = -g \circ h \geq 0$ ($h < 0$) and therefore $g \circ h \leq 0$. $|g| = g, |h| = -h$. This implies $|g| \circ |h| = g \circ (-h) = g \circ h = |g \circ h|$. If $g < 0$ then $|g \circ h| = g \circ h$ and $|g| \circ |h| = (-g) \circ (-h) = g \circ h = |g \circ h|$.

To (25):

(i) Let g be > 0 . $\text{sign } \bar{g} = \text{sign } g$ implies $\bar{g} > 0, g > 0, h' := -h > 0$. $|g| \circ |h| = g \circ (-h) = g \circ h'$. Therefore $|g \circ h| = |(\bar{g} + g) \circ h| \leq |\bar{g} \circ h| + |g \circ h| = |\bar{g} \circ (-h')| + |g \circ (-h')| = |\bar{g} \circ h'| + |-g \circ h'| = \bar{g} \circ h' + g \circ h' = (\bar{g} + g) \circ h' = g \circ h' = |g| \circ |h|$, and (25) is proved in this case.

(ii) $g = 0$ trivially implies (25).

(iii) $g < 0$ implies $\bar{g} < 0, g < 0, g' := -g > 0$. (25) (i) tells us that $|g \circ h| = |(-g') \circ h| = |-(g' \circ h)| = |g' \circ h| \leq |g'| \circ |h| = |g| \circ |h|$.

To (26):

(i) If $\bar{g} = 0$ or $g = 0$, then (26) is valid because of (24).

(ii) Let $\bar{g}$ be > 0 , consequently $g < 0, g := -\bar{g} > 0$. Then $|g \circ h| = |g \circ (-h')| = |(\bar{g} + g) \circ (-h')| = |\bar{g} \circ h' + g' \circ h'| = |\bar{g} \circ h'| + |g' \circ h'| = \bar{g} \circ h' + g' \circ h' = (\bar{g} + g') \circ h' = g \circ h' = |g \circ h|$. On the other hand, $|g| \circ |h| = |\bar{g} - g'| \circ |h| \leq |\bar{g}| \circ |h'| + |g'| \circ |-h'| = \bar{g} \circ h' + g' \circ h' = |g \circ h|$. This verifies (26).

(iii) Let $\bar{g}$ be < 0 , then $g > 0$ and $\bar{g}' := -\bar{g} > 0, g' := -g = -\bar{g} - \underline{g} = \bar{g}' + \underline{g}'$, with $\bar{g}' > 0, \underline{g}' < 0$. Applying (26) (ii) one arrives at

$$|g \circ h| = |-g \circ h| = |g' \circ h| \geq |g'| \circ |h| = |g| \circ |h|.$$

To show the remark, consider again the polynomial ring $R[x]$ with the lexicographic order. At first we take

$$g: = x^2 + x - 1 = (x^2 - 1) + x,$$

$$h: = -x - 1 < 0, h \notin C(R[x]) = R. \text{ We have } |g \circ h| = |(x^2 + x - 1) \circ (-x - 1)| = |x^2 + x - 1| = x^2 + x - 1, \text{ but}$$

$$|g| \circ |h| = (x^2 + x - 1) \circ (x + 1) = x^2 + 3x + 1,$$

therefore $|g \circ h| < |g| \circ |h|$. Then we take

$$g: = x^2 - x + 1 = (x^2 + 1) - x > 0, \quad h: = -x + 1 < 0; h \notin R.$$

$$|g \circ h| = |(x^2 - x + 1) \circ (-x - 1)| = |x^2 - x - 1| = x^2 - x + 1,$$

but

$$|g| \circ |h| = (x^2 - x + 1) \circ (x - 1) = x^2 - 3x + 3, \text{ therefore}$$

$$|g \circ h| > |g| \circ |h|, \text{ and theorem 5 is completely proved.}$$

Specializing $G = R[x]$, theorem 5 can be formulated in this way:

$$|g \circ h| = |g| \circ |h|, \text{ if } h \geq 0 \text{ or } g \text{ contains only even or odd degrees.}$$

$$|g \circ h| \leq |g| \circ |h|, \text{ if } h < 0 \text{ and the coefficients of the greatest even and odd degrees have the same sign.}$$

$$|g \circ h| \geq |g| \circ |h|, \text{ if } h < 0 \text{ and the coefficients of the greatest even and odd degrees have opposite signs.}$$

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