One particular class of Eulerian numbers of higher order and some allied sequences of numbers

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Introduction

Some number-theoretic investigations commenced in [1] and shortly continued on the begining of the present paper led to one particular class of Eulerian numbers of higher order and some new sequences of numbers. It seems that such natural development of the theory shows promise of their fertility. In all these considerations the main role has a number-theoretic function introduced in [7] and designated by $s_r(n)$. The present paper is only one part of the attempt to find a unified approach to some at the first sight heterogeneous unsolved problems of number theory.

1. One particular class of Eulerian numbers of higher order

Definition 1. Let a and b be arbitrary numbers. We define numbers

$$A_{2k}(a, b) = \frac{a^{2k}}{2k+1} C'_{k+1}(c) \tag{1}$$

where $C_m(t)$ and c are defined in reference [1] by relations (3) and (10) respectively.[1])

Because of $C_1(t)=t$, it results that for every $a\neq 0$ and b $A_0(a, b)=1$. Having knowledge of the connection between polynomials $C_m(t)$ and Bernoulli polynomials, cfr. [1] relation (21) with + sign, we can express numbers $A_{2k}(a, b)$ also by means of Bernoulli polynomials

$$A_{2k}(a, b) = \frac{2a^{2k+1}}{(2k+1)(a+2b)} B_{2k+1}(1+b/a), \tag{2}$$

$k=0, 1, 2, \ldots$. Generating function for numbers $A_{2k}(a, b)$ we shall obtain easily

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1) I.e.

$$C_m(t) = \sum_{k=1}^{m} \alpha(m, k) t^k \qquad (m = 1, 2, 3, \ldots),$$

where

$$\alpha(m, k) = \frac{2^{k-1}}{k} \sum_{s=0}^{2m-2k} \binom{2m-1}{s} \binom{2m-s-k-1}{k-1} B_s \qquad (k = 1, 2, \ldots, m)$$

moreover

$$c = b(a+b)/2a^2.$$

starting from the similar formula for Bernoulli polynomials [2], p. 36, formula 1. 13 (2). From the latter it is possible to derive

$$\sum_{n=0}^{\infty} B_{2n}(x) z^{2n}/(2n)! = \frac{z}{2} \frac{\operatorname{ch}(x-1/2)z}{\operatorname{sh}(z/2)},$$

whence integrating with respect to x from 0 to y we find

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}(y)}{2n+1} \frac{z^{2n}}{(2n)!} = \frac{\operatorname{sh}(y-1/2)z}{2\operatorname{sh}(z/2)}.$$

Substituting here az instead of z, $y = 1+b/a$, and making some adaptations dictated by the formula (2), we arrive at the proposed generating function

$$\sum_{k=0}^{\infty} A_{2k}(a, b) \frac{z^{2k}}{(2k)!} = \frac{a \operatorname{sh}(a+2b)z/2}{(a+2b)\operatorname{sh}(az/2)}. \tag{3}$$

Some number-theoretic properties of numbers $A_{2k}(a, b)$, described below, were the primary reason for Definition 1.

Theorem 1. *Let a and b be integers, $a \neq 0$, m and r natural numbers, and suppose there exists a natural number $q|r$ and for every primedivisor p of q, $p \geqq 2m-1$. Then*

$$\sum_{n=1}^{r} (an+b)^{2m-1} \equiv (2m-1) v A_{2m-2}(a, b) \pmod{q^2}, \tag{4}$$

where v is defined by (9), [1].

PROOF. Develope the polynomial $C_m(v/a+c)$, that stands on the right-hand side of formula (8), [1], in Taylor series at the point c:

$$\sum_{r=1}^{r} (an+b)^{2m-1} = a^{2m-1} \left[\frac{v}{a} C'_m(c) + \frac{v^2}{2!a^2} C''_m(c) + \ldots + \frac{v^m}{m!a^m} C^{(m)}_m(c)\right].$$

The expression on the right-hand side is a polynomial in a. Further it is obvious that $q|v$, and with regard to Lemma 1, reference [1], from the last equation the following congruence results:

$$\sum_{n=1}^{r} (an+b)^{2m+1} \equiv a^{2m-2} v C'_m(c) \pmod{q^2}.$$

Herefrom by Definition 1 we obtain congruence (4).

In this paper we shall repeatedly use the so called symbolic method, rigorously founded for finite cases by BLISSARD and LUCAS (short account on this may be found in [3], p. 250), however transition to infinite cases does not introduce essential novelties. Instead of $A_{2k}(a, b)$ we shall write shortly A_{2k} whenever it should not cause ambiguity. With this convention in mind the relation (3) is rewritten in symbolic form

$$(a+2b) \operatorname{sh} \frac{az}{2} \operatorname{ch} Az = a \operatorname{sh}(a+2b) \frac{z}{2}. \tag{5}$$

If we add $A_{2k+1}=0$ for $k=0, 1, 2, \ldots$ to the Definition 1, then $\operatorname{sh} Az=0$ and by equation (5)

$$(a+2b)\operatorname{sh}(A+a/2)z = a\operatorname{sh}(a+2b)\frac{z}{2}.$$

Comparing the coefficients of the equal powers of z, we obtain recurrence formula for numbers A_{2k}:

$$(A+a/2)^{2k+1} = a(a+2b)^{2k}/2^{2k+1}, \qquad k = 0, 1, 2, \ldots. \tag{6}$$

Theorem 2. *Let a and b be integers, $a\neq 0$, $(a, b)=1$, p an odd prime. $A_{p-1}(a, b)$ is integral* (mod p) *if and only if $p\nmid a(a+2b)$.*

PROOF. From relation (6) we conclude that A_{2s} is integral (mod p) for every $s < (p-1)/2$. From the same relation we obtain

$$pA_{p-1} \equiv (a+2b)^{p-1} - a^{p-1} \pmod{p}.$$

If A_{p-1} is integral (mod p), then from supposition $p|a$ follows $p|a+2b$, hence $p|b$, that contradicts the hypothesis $(a, b)=1$. An analogous reasoning shows that $p\nmid a+2b$. Contrary, assuming that $p\nmid a(a+2b)$, from the above congruence it follows by Fermat's theorem that $pA_{p-1}\equiv 0 \pmod{p}$, hence A_{p-1} is integral (mod p). This proves the theorem.

An interesting application of the just proved theorem starts from congruence (4), where we shall put $2m-1 = p$, p a prime, $r=kp$, k natural number, $q=p$. Then

$$\sum_{n=1}^{kp} (an+b)^p \equiv 0 \pmod{p^2} \quad \text{if} \quad p\nmid a(a+2b).$$

On the base of Theorem 2 we can also state a necessary condition that all numbers $A_{2k}(a, b)$, $k=0, 1, 2, \ldots$, be integral: for every odd prime p there holds $p\nmid a(a+2b)$. It will be fulfilled if $a=2^r$ and $a+2b = 2^m$ whence $b = 2^{m-1}-2^{r-1}$. Now we must distinguish three cases:

1. $r=m$. Then $b=0$, and from relation (2) $A_{2k}(a, 0)=0$, $k=1, 2, \ldots$.

2. $r>m$. Owing to the condition $(a, b)=1$ it is necessary to take $m=1$, hence $a=2^r$, $b=-(2^{r-1}-1)$, $r=1, 2, 3, \ldots$.

3. $r<m$. In this case to satisfy condition $(a, b)=1$ we must assume $r=1$, $a=2$, $b=2^{m-1}-1$. In view of the relation (2) and the connection between Bernoulli polynomials and the sum of powers of natural numbers we find that

$$A_{2k}(2, 2^{m-1}-1) = \frac{1}{2^{m-2}} \sum_{s=1}^{2^{m-2}} (2s-1)^{2k}. \tag{7}$$

By the induction on m it is easily proved that the expression on the right-hand side of (7) is integer for every $k\geqq 0$.

In this paper we shall investigate only the second of the three mentioned cases. To simplify notation, we shall r replace by $r+1$ and introduce designation

$$4R=2^r, \qquad r=0, 1, 2, \ldots. \tag{8}$$

For numbers $A_{2k}(8R, 1-4R)$ we use notation $M_{2k}^{(r)}$, where superscript (r) will be

omitted whenever it would be possible, especially in symbolic relations, in which $(M^{(r)})^{2k}$ and M^{2k} will posses the same meaning as $M_{2k}^{(r)}$. Now relation (2) gives

$$M_{2k}^{(r)} = -\frac{2^{(r+1)(2k+1)}}{2k+1} B_{2k+1}(1/2-1/8R), \tag{9}$$

$k=0, 1, 2, \ldots$. Herefrom we see that $M_{2k}^{(0)}=0$, $k=0, 1, 2, \ldots$, so there remained cases $r=1, 2, 3, \ldots$. If $r=1$, from [4], formula (43*), p. 29, we conclude that $M_{2k}^{(1)}=$ $=E_{2k}$, Eulerian numbers. As concerns the remaining sequences $M_{2k}^{(r)}$, there holds

Theorem 3. *Numbers $M_{2k}^{(r)}$, $r=2, 3, 4, \ldots$, represent a class of Eulerian numbers of higher order. More precisely, according to the definition of the last ones*, cfr. [2], 1. 15 (21), p. 43,

$$\sum_{n=0}^{\infty} E_n^{(m)}(a_1, a_2, \ldots, a_m) z^n/n! = [\operatorname{ch}(a_1 z) \operatorname{ch}(a_2 z) \ldots \operatorname{ch}(a_m z)]^{-1}, \tag{10}$$

numbers $M_{2k}^{(r)}$ correspond to that class of Eulerian numbers of higher order, for which $a_1=1$, $a_2=2, \ldots, a_k=2^{k-1}, \ldots, a_m=2R$.

Proof. Substituting $a=8R$, $b=1-4R$ in relation (3) we obtain the generating function for numbers $M_{2k}^{(r)}$

$$\sum_{k=0}^{\infty} M_{2k}^{(r)} z^{2k}/(2k)! = \frac{4R \operatorname{sh} z}{\operatorname{sh} 4Rz}, \tag{11}$$

where the expression on the right-hand side may take the form [ch z ch $2z$ ch $4z$... ... ch $2Rz$]$^{-1}$. Function on the right-hand side of definition (10) is even, hence $E_{2k+1}^{(m)}(a_1, a_2, \ldots, a_m)=0$ for $k=0, 1, 2, \ldots$. Comparing relations (10) and (11) there follows statement of the theorem.

By use of the relation (6) we obtain recurrence formula for numbers $M_{2k}^{(r)}$:

$$(4R+M)^{2m+1} = 4R, \tag{12}$$

$m=0, 1, 2, \ldots$, $r=1, 2, 3, \ldots$, which is possible to write in the form

$$(4R+M)^{2m+1}+(4R-M)^{2m+1} = 8R. \tag{12a}$$

These formulas are suitable for calculation of numbers $M_{2k}^{(r)}$. However, numbers $M_{2k}^{(r)}$ satisfy another recurrence formula, which will be of greater interest in further investigations. Modifying the right-hand side of the relation (11) according to the known formula

$$\operatorname{sh} 4Rz/\operatorname{sh} z = 2 \sum_{k=1}^{2R} \operatorname{ch}(2k-1)z,$$

(see for instance [5], formula (420), p. 78), we get the symbolic relation

$$\sum_{k=1}^{2R} [\operatorname{ch}(M+2k-1)z + \operatorname{ch}(M-2k+1)z] = 4R.$$

Comparing the coefficients of the equal powers of z, recurrence formula follows

$$\sum_{k=1}^{2R} [(M+2k-1)^{2m}+(M-2k+1)^{2m}] = 4R\delta_{0,m}, \tag{13}$$

where $\delta_{0,m}$ is Kronecker's symbol, $m=0, 1, 2, \ldots$, $r=1, 2, 3, \ldots$.

Theorem 4. *All numbers* $M_{2k}^{(r)}$, $r=1, 2, 3, \ldots$, $k=0, 1, 2, \ldots$, *are odd integers.*

PROOF. By induction we shall prove that numbers M are integers. First, from relation (13) it is obvious that $M_0^{(r)}=1$ for 1, 2, 3, ... and secondly, suppose the hypothesis is true for $k = 0, 1, 2, \ldots, m-1$, and from the same relation (13) calculate $M_{2m}^{(r)}$

$$2RM_m^{(r)} = -\sum_{s=0}^{m-1} \binom{2m}{2s} M_{2s}^{(r)} \sum_{k=1}^{2R} (2k-1)^{2m-2s}.$$

If we remember what has been said in connection with the expression (7), it is now concluded that the inner sum is divisible by $2R$ for $s = 0, 1, 2, \ldots, m-1$, hence $M_{2m}^{(r)}$ is also integer. Thus induction proof is finished. Take residues of the left and right-hand sides of the equation (12) modulo $(4R)^3$, after cancelation by $4R$ follows

$$(2m+1)M_{2m}^{(r)} \equiv 1 \pmod{(4R)^2}, \tag{14}$$

$m=0, 1, 2, \ldots, r=1, 2, 3, \ldots$. Now what we need to prove to complete proof of the theorem is only a special case of congruence (14).

Theorem 5. $(-1)^k M_{2k}^{(r)} > 0, \quad k = 0, 1, 2, \ldots, \; r = 1, 2, 3, \ldots.$

PROOF. This theorem is consequence of the relation (9) and of the flow of the function $B_{2k+1}(x)$ in interval $1/4 \leqq x < 1/2$, cfr. [4], pp. 22—23.

2. Some sequences of numbers generated by sequences $M_{2k}^{(r)}$

In this paragraph it will be shown how to every sequence $M_{2k}^{(r)}$ one can associate R sequence of numbers with even subscripts and as much again sequences of numbers with odd subscripts. Definitions that will be laid down in this connection, may appear quite arbitrary. On the contrary, they are consequences of fairly ample investigations of numerous particular cases. The principal idea for these investigations rests upon already observed facts. Namely, already de Moivre has observed (cfr. [6], pp. 7—8) that in the expression for (finite) sum of powers of natural numbers as well as in the expression for (infinite) sum of reciprocal even powers of natural numbers appear Bernoulli numbers. In a similar manner in the expression for alternating (finite) sum of powers of odd numbers as well as in the expression for alternating (infinite) sum of reciprocal odd powers of odd numbers appear Eulerian numbers. Whether the similar circumstances will repeat with other related cases is by no means trivial question to which we want to give an answer in the subsequent paragraphs.

Recurrence formula (13) can be written thus:

$$\sum_{k=1}^{4R} (M+2k-1-4R)^m = 4R\delta_{0,m}. \tag{15}$$

Now let $f(x)$ be any polynomial in x, and develope the function $f(x+M+2k-1-4R)$ in Taylor series at the point x: $f(x+M+2k-1-4R) = f(x)+(M+2k-1-4R)f'(x)/1!+(M+2k-1-4R)^2 f''(x)/2!+\ldots$. Performing summation

of the left and right-hand sides with respect to k from 1 to $4R$, in view of the relation (15) we obtain

$$\sum_{k=1}^{4R} f(x+M+2k-1-4R) = 4Rf(x), \tag{16}$$

$r=1, 2, 3, \ldots$, which is the fundamental symbolic formula for numbers M. In the next place substitute $2t-1$ instead of x in the relation (16), multiply this relation by $s_r(t+w)$, where w is arbitrary integer and $s_r(n)$ is the number-theoretic function defined in [7] thus

$$s_r(n) = \begin{cases} +1 & \text{for} \quad n \equiv 1, 2, 3, \ldots, 2R \pmod{4R} \\ -1 & \text{for} \quad n \equiv 2R+1, 2R+2, \ldots, 4R \pmod{4R}, \end{cases}$$

$r=1, 2, 3, \ldots$, and carry out summation on t from 1 to q:

$$4R \sum_{t=1}^{q} s_r(t+w) f(2t-1) = \sum_{k=1}^{4R} \sum_{t=1}^{q} s_r(t+w) f(2t+2k+M-2-4R). \tag{17}$$

The inner sum on the right-hand side can be written in the form

$$\sum_{t=k+1}^{q+k} s_r(t-k+w) f(2t+M-4R-2),$$

and then decomposed into three sums $U_1(k)$, $U_2(k)$, $U_3(k)$ determined by the ranges of summation $k+1 \leqq t \leqq 4R+1$; $4R+2 \leqq t \leqq q$; $q+1 \leqq t \leqq q+k$, respectively, provided that

$$q \geqq 4R+1. \tag{18}$$

Substituting these in the right-hand side of the equation (17) we obtain three double sums in which we shall change the order of summation

$$\sum_{k=1}^{4R} U_1(k) = \sum_{t=1}^{4R-1} f(2t+M-4R) \sum_{k=1}^{t} s_r(t-k+w+1),$$

$$\sum_{k=1}^{4R} U_2(k) = \sum_{t=4R+2}^{q} f(2t+M-4R-2) \sum_{k=1}^{4R} s_r(t-k+w) = 0,$$

$$\sum_{k=1}^{4R} U_3(k) = -\sum_{t=1}^{4R-1} f(2t+2q+M-4R) \sum_{k=1}^{t} s_r(t+q-k+w+1).$$

In this manner the relation (17) now reads

$$4R \sum_{t=1}^{q} s_r(t+w) f(2t-1) = \sum_{t=1}^{4R-1} \sum_{k=1}^{t} \{s_r(t-k+w+1) f(2t+M-4R) - \tag{19}$$

$$- s_r(t+q-k+w+1) f(2t+2q+M-4R),$$

recalling that condition (18) must be fulfilled. This result leads us to the following two definitions corresponding to the particular cases of polynomial $f(x)$, namely $f(x)=x^{2m}$ and $f(x)=x^{2m+1}$ respectively.

Definition 2.

$$n(w)N_{2m}^{(r)}(w) = \sum_{t=1}^{4R-1} (2t+M-4R)^{2m} \sum_{k=1}^{t} s_r(t-k+w+1), \tag{20}$$

where $m=0, 1, 2, \ldots$, $r=2, 3, 4, \ldots$, for w see later.

Definition 3.

$$l(w)L_{2m+1}^{(r)}(w) = \sum_{t=1}^{4R-1} (2t+M-4R)^{2m+1} \sum_{k=1}^{t} s_r(t-k+w+1), \tag{21}$$

where $m=0, 1, 2, \ldots$, $r=2, 3, 4, \ldots$, for w see later.

The factors $n(w)$ and $l(w)$ were introduced to attain standardization: $N_0^{(r)}(w)=1$ and $L_1^{(r)}(w)=1$ for every r and w. Now it is obvious that functions $n(w)$ and $l(w)$ must be periodic with the same period that the function $s_r(n)$ has, namely $4R$. However these functions have also the property that for every integer k relations $n(k+2R) = -n(k)$ and $l(k+2R) = -l(k)$ hold. Hence it is sufficient to determine these functions within a half-period. From relations (20) and (21) for $m=0$, after some calculation one can obtain

$$n(w) = 4R(R-w), \qquad w=0, 1, 2, \ldots, 2R-1, \tag{23}$$

$$l(w) = 4Rw(w-2R), \qquad w=1, 2, 3, \ldots, 2R. \tag{24}$$

3. Generating function and recurrence formula for numbers $N_{2m}^{(r)}(w)$

Instead of notation $N_{2m}^{(r)}(w)$ we shall use designation $N_{2m}(w)$ or simply N_{2m} whenever it could not cause misunderstanding. Relation (20) we can write in the symbolic form

$$n(w)\operatorname{ch} Nz = \sum_{t=1}^{4R-1} \operatorname{ch}(M-4R+2t)z \sum_{k=1}^{t} s_r(t-k+w+1). \tag{25}$$

It is done to make possible elimination of numbers $M_{2m}^{(r)}$ by use of the formula (11) which we shall brought also in the symbolic form

$$\operatorname{ch} Mz = 4R \operatorname{sh} z / \operatorname{sh} 4Rz. \tag{26}$$

Since $\operatorname{ch}(M-4R+2t)z = \operatorname{ch} Mz \operatorname{ch}(4R-2t)z$, from relations (25), (26) and (23), after some intricate calculation (which will be omitted for the sake of conciseness) one can finally arrive at the proposed generating function

$$\operatorname{ch} Nz = \frac{1}{(R-w)\operatorname{ch} 2Rz} \sum_{s=1}^{R-w}{}' \operatorname{ch}(2s-1)z, \tag{27}$$

where $w=0, 1, 2, \ldots, R-1$. Herefrom we read out the recurrence formula for numbers $N_{2m}^{(r)}(w)$

$$(N+2R)^{2m} = \frac{1}{R-w} \sum_{s=1}^{R-w}{}' (2s-1)^{2m}, \tag{28}$$

where $m=0, 1, 2, \ldots, w=0, 1, 2, \ldots, R-1, r=2, 3, 4, \ldots, N^{(r)}_{2k+1}(w)=0$ for $k=0, 1, 2, \ldots$.

If superscript r is fixed, then to the sequence $M^{(r)}_{2m}$ there are associated R sequences of numbers $N^{(r)}_{2m}(w)$, which have even subscripts. Not all numbers N are integers.

Theorem 6. *Numbers $N^{(r)}_{2m}(w)$ are integers for $w = R-2^n$, $n=0, 1, 2, \ldots, r-2$; $r=2, 3, 4, \ldots$.*

PROOF. On the base of what has been said in connection with expression (7), the right-hand side of (28) represents integer if $w = R-2^n$. In these cases the theorem can be proved by induction without much trouble, using formula (28).

Of course, numbers $(R-w)N^{(r)}_{2m}(w)$ are always integers.

In this paragraph we shall mention also the Mittag—Lefler's developement of the generating function ch Nz and a relation which results herefrom.

The first reads:

$$\operatorname{ch} Nz = \frac{1}{R(R-w)} \sum_{t=1}^{\infty} (-1)^{t-1} \sum_{s=1}^{R-w} \cos(2s-1)(2t-1)\pi/4R \cdot \cdot \frac{(2t-1)\pi/4R}{z^2+((2t-1)\pi/4R)^2}. \tag{29}$$

Develope expression on the right-hand side in Taylor's series with respect to z at the point $z=0$, apply Cauchy's theorem about double series, then comparison of coefficients by the equal powers of z yields

$$\sum_{t=1}^{\infty} (-1)^{t-1} \sum_{s=1}^{R-w} \cos(2s-1)(2t-1)\pi/4R \frac{1}{(2t-1)^{2m+1}} = = \frac{(-1)^m \pi^{2m+1} R(R-w) N^{(r)}_{2m}(w)}{(2m)!\,(4R)^{2m+1}}, \tag{30}$$

where $m=0, 1, 2, \ldots, w=0, 1, 2, \ldots, R-1, r=2, 3, 4, \ldots$. Coefficient of $1/(2t-1)^{2m+1}$ in the series on the left-hand side of the equation (30) designate by $S(t, w)$. It can be shown that

$$S(t, w) = \frac{\cos(2t-1)w\pi/2R}{2\sin(2t-1)\pi/4R}. \tag{31}$$

4. Generating function and recurrence formula for numbers $L^{(r)}_{2m+1}(w)$

Similarly as in the preceding paragraph, we shall use simplified notations $L^{(r)}_{2m+1}$ and L_{2m+1} instead of $L^{(r)}_{2m+1}(w)$, particularly in symbolic formulas. Rewrite the relation (21) in symbolic form:

$$l(w) \operatorname{sh} Lz \sum_{t=1}^{4R-1} \operatorname{sh}(2t+M-4R)z \sum_{k=1}^{t} s_r(t-k+w+1). \tag{32}$$

Owing to symbolic identity $\operatorname{sh}(M+2t-4R) = \operatorname{ch} Mz \operatorname{sh}(2t-4R)z$, by means of (24) and (26) relation (32) becomes

$$w(2R-w)\operatorname{sh} Lz = \frac{\operatorname{sh} z}{\operatorname{sh} 4Rz} \sum_{t=1}^{4R} \operatorname{sh}(4R-2t)z \sum_{k=1}^{t} s_r(t-k+w+1).$$

Now we shall omit not so short discussion on the transformation of the right member. The final result is the generating function:

$$\operatorname{sh} Lz = \frac{1}{w(2R-w)\operatorname{ch} 2Rz} \sum_{s=R-w+1}^{R} \operatorname{sh}(2s-1)z, \tag{33}$$

where $w=1, 2, 3, \ldots, R$. Herefrom we obtain recurrence formula for numbers $L_{2m+1}^{(r)}(w)$

$$(L+2R)^{2m+1} = \frac{1}{w(2R-w)} \sum_{s=R-w+1}^{R} (2s-1)^{2m+1}, \tag{34}$$

where $m=0, 1, 2, \ldots$, $w=1, 2, 3, \ldots, R$, $r=2, 3, 4, \ldots$, $L_{2k}^{(r)}(w)=0$ for $k=0, 1, 2, \ldots$. Thus to every sequence $M_{2m}^{(r)}$ with fixed r there correspond R sequences of numbers $L_{2m+1}^{(r)}(w)$. There is a remarkable difference between numbers N and L. With this fact deals

Theorem 7. *All numbers $L_{2m+1}^{(r)}(w)$ are odd integers.*

PROOF. If we prove that right member of (34) is always integer, then the same fact for numbers L is easily proved by induction. The sum on the right-hand side of equation (34) we shall designate by S. The proof is divided into three parts:

a) Assume that p is odd prime and $p^n|2R-w$. The end terms of the sum S are $(2R-2w+1)^{2m+1}$ and $(2R-1)^{2m+1}$. If w is odd, middle term of S is $(2R-w)^{2m+1}$ and then it is evidently

$$S \equiv \sum_{i=\frac{1-w}{2}}^{\frac{w-1}{2}} (2i)^{2m+1} \equiv 0 \pmod{p^n}.$$

If w is even $(2R-w)^{2m+1}$ is not term of the sum at all, and then

$$S \equiv \sum_{i=1-\frac{w}{2}}^{\frac{w}{2}} (2i-1)^{2m+1} \equiv 0 \pmod{p^n}.$$

b) Assume that p is odd prime and $p^n|w$. In the sum S there are w terms. Let $w=qp^n$, $p^n = 2k+1$. Any sequence of $2k+1$ succesive odd numbers form a complete system of residues modulo $2k+1$. Hence that system of residues we can always bring into the form: $-2k, -2(k-1), \ldots, -2, 0, 2, \ldots, 2(k-1), 2k$. On this base we have

$$S \equiv q \sum_{i=-k}^{k} (2i)^{2m+1} \equiv 0 \pmod{p^n}.$$

c) Now suppose that $n \geqq 1$ is the largest number such that $2^n | w$. Then $2^n | 2R-w$ because $w \leqq R$. Hence, let $w=2^n q$, $2 \nmid q$. We start this part of proof with slightly modified expression for S:

$$S = \sum_{i=1-\frac{w}{2}}^{\frac{w}{2}} (2R-w+2i-1)^{2m+1},$$

whence, assuming $m>0$, it follows (after some calculations)

$$S \equiv 2(2m+1)(2R-q2^n) \sum_{i=1}^{w} (2i-1)^{2m} \pmod{2^{3n}}. \tag{35}$$

On this point it would be possible to prove that numbers L are integers, but we need more, therefore continue with application of a well-known formula

$$\sum_{i=1}^{\frac{w}{2}} (2i-1)^{2m} = \frac{1}{2m+1}\left[B_{2m+1}(w) - 2^{2m} B_{2m+1}\left(\frac{w}{2}\right)\right].$$

Here, without much labor, one can establish congruences

$$B_{2m+1}(w) \equiv (2m+1) w B_{2m} \pmod{2^n}$$

$$2^{2m} B_{2m+1}(w/2) \equiv 0 \pmod{2^n}, \qquad m \geqq 1,\ n \geqq 1,$$

so we find

$$\sum_{i=1}^{\frac{w}{2}} (2i-1)^{2m} \equiv w B_{2m} \pmod{2^n},$$

and with respect to congruence (35)

$$S \equiv 2^{2n}(2m+1) q 2 B_{2m} \pmod{2^{2n+1}}.$$

In the first place we conclude that S is divisible by 2^{2n} (which completes the proof that L are integers), and afterwards

$$S/2^{2n} \equiv (2m+1) q 2 B_{2m} \equiv 1 \pmod 2.$$

In view of the above result, the relation (34) implies

$$L^{(r)}_{2m+1}(w) \equiv 1 \pmod 2.$$

The proof is completed.

At last we shall mention the Mittag—Leffler's developement of the generating function:

$$\operatorname{sh} Lz = \frac{1}{w(2R-w)R} \sum_{t=1}^{\infty} \left[(-1)^{t-1} \sum_{s=R-w+1}^{R} \sin(2s-1)(2t-1)\pi/4R\right] \cdot \tag{36}$$

$$\cdot \frac{z}{z^2 + ((2t-1)\pi/4R)^2}$$

If one develope the function on the right-hand side in Taylor's series at point $z=0$, apply Cauchy's theorem on double series, then comparison of coefficients gives

$$\sum_{t=1}^{\infty}\left((-1)^{t-1}\sum_{s=R-w+1}^{R}\sin(2s-1)(2t-1)\pi/4R\right)\frac{1}{(2t-1)^{2m}}= \tag{37}$$
$$=\frac{(-1)^{m-1}\pi^{2m}Rw(2R-w)L_{2m-1}^{(r)}(w)}{(2m-1)!(4R)^{2m}},$$

where $m=1, 2, 3, \ldots$, $w=1, 2, \ldots, R$, $r=2, 3, 4, \ldots$. Designate by $S_1(t, w)$ the coefficient of $1/(2t-1)^{2m}$ in the above sum. It can be modified thus

$$S_1(t, w)=\frac{\cos(2t-1)u\pi/2R}{2\sin(2t-1)\pi/4R}, \tag{38}$$

where substitution $w=R-u$ is made.

5. Some finite and infinite series whose summation is connected with numbers N and L

In the previous paragraphs the numbers $N_{2m}^{(r)}(R)$ and $L_{2m+1}^{(r)}(0)$ were not defined. Their definition may be realised through a limit process, but it is not of interest for this research. However, for our purposes it is necessary to have detailed knowledge of numbers $n(w)N_{2m}^{(r)}(w)$ and $l(w)L_{2m+1}^{(r)}(w)$ for arbitrary integer w. In this connection it is easily verified that $n(R)N_{2m}^{(r)}(R)=0$ and $l(0)L_{2m+1}^{(r)}(0)=0$, and also

$$n(w+2R)N_{2m}^{(r)}(w+2R)=-n(w)N_{2m}^{(r)}(w)$$

and

$$l(w+2R)L_{2m+1}^{(r)}(w+2R)=-l(w)L_{2m+1}^{(r)}(w)$$

for $m=0, 1, 2, \ldots$, $r=2, 3, 4, \ldots$, and arbitrary integer w. Now we are able to consider again the formula (19), which after substitution $f(x)=x^{2m}$ and application of definitions (20) and (21) becomes

$$4R\sum_{t=1}^{q}s_r(t+w)(2t-1)^{2m}=n(w)N_{2m}^{(r)}(w)-n(w+q)N_{2m}^{(r)}(w+q)-$$
$$-n(w+q)\sum_{s=0}^{m-1}\binom{2m}{2s}N_{2s}^{(r)}(w+q)(2q)^{2m-2s}- \tag{39}$$
$$-l(w+q)\sum_{s=0}^{m-1}\binom{2m}{2s+1}L_{2s+1}^{(r)}(w+q)(2q)^{2m-2s-1},$$

provided that (18) is fulfilled.

Herefrom we infer that

$$4R\sum_{t=1}^{q}s_r(t+w)(2t-1)^{2m}\equiv n(w)N_{2m}^{(r)}(w)-n(w+q)N_{2m}^{(r)}(w+q)\pmod{2q}, \tag{40}$$

where $q \equiv 4R+1$, $m=0, 1, 2, \ldots$, $r=2, 3, 4, \ldots$, w arbitrary integer. Similarly if we put $f(x)=x^{2m+1}$, relation (19) amounts to

$$4R \sum_{t=1}^{q} s_r(t+w)(2t-1)^{2m+1} = l(w) L_{2m+1}^{(r)}(w) - l(w+q) L_{2m+1}^{(r)}(w+q) -$$

$$-n(w+q) \sum_{s=0}^{m} \binom{2m+1}{2s} N_{2s}^{(r)}(w+q)(2q)^{2m-2s+1} - \tag{41}$$

$$-l(w+q) \sum_{s=0}^{m-1} \binom{2m+1}{2s+1} L_{2s+1}^{(r)}(w+q)(2q)^{2m-2s},$$

again with restriction (18). An immediate consequence of this relation is the congruence

$$4R \sum_{t=1}^{q} s_r(t+w)(2t-1)^{2m+1} \equiv l(w) L_{2m+1}^{(r)}(w) - l(w+q) L_{2m+1}^{(r)}(w+q) \pmod{2q}, \tag{42}$$

where $q \equiv 4R+1$, $m = 0, 1, 2, \cdots$, $r = 2, 3, 4, \cdots$, w arbitrary interger.

Also for present and some later investigations will be useful

Definition 4.

$$K_r(2m+1) = \sum_{t=1}^{\infty} \frac{s_r(t)}{(2t-1)^{2m+1}}, \quad L_r(2m) = \sum_{t=1}^{\infty} \frac{(-1)^{t-1} s_r(t)}{(2t-1)^{2m}}, \quad r = 2, 3, 4, \ldots. \tag{43}$$

Remark. It is to be distinguished between designations $L_{2m+1}^{(r)}(w)$ and $L_r(2m)$. The latter will not be subject to symbolic method.

Definition 5.

$$K(s) = \sum_{t=1}^{\infty} \frac{1}{(2t-1)^s}, \quad \operatorname{Re} s > 1; \qquad L(s) = \sum_{t=1}^{\infty} \frac{(-1)^{t-1}}{(2t-1)^s}, \quad \operatorname{Re} s \geqq 1; \tag{44}$$

where s is the complex variable. Obviously

$$\lim_{r \to \infty} K_r(2m+1) = K(2m+1) \quad \text{and} \quad \lim_{r \to \infty} L_r(2m) = L(2m) \quad \text{for} \quad m = 1, 2, 3, \ldots.$$

It is now interesting that the expression for sums of series (43) involve the numbers N and L respectively. It was already shown how coefficients $S(t, w)$, given by (31), appear in the left member of (30). These coefficients appear also in the relation (11), reference [7], and if we take into consideration all mentioned, after summation with respect to w the following equation results

$$K_r(2m+1) = \frac{(-1)^m \pi^{2m+1}}{(2m)!(4R)^{2m+1}} \sum_{w=0}^{R-1} \frac{(\delta_{0,w}-2) \sin \pi/4R \cos w\pi/2R}{\sin (2w-1)\pi/4R \sin (2w+1)\pi/4R} (R-w) N_{2m}^{(r)}(w), \tag{45}$$

$$m = 0, 1, 2, \ldots, \quad r = 2, 3, 4, \ldots.$$

Similarly by means of relations (37) and (38) of this paper and (11) of reference [7], after summation with respect to u we obtain

$$(46)\qquad \begin{aligned} I_r(2m) = {} & \frac{(-1)^{m-1}\pi^{2m}}{(2m-1)!(4R)^{2m}} \sum_{u=0}^{R-1} \frac{(\delta_{0,u}-2)\sin \pi/4R \cos u\pi/2R}{\sin(2u-1)\pi/4R \sin(2u+1)\pi/4R} \cdot \\ & \cdot (R^2-u^2) L_{2m-1}^{(r)}(R-u), \qquad m = 1,2,3,\ldots, \quad r = 2,3,4,\ldots . \end{aligned}$$

The relations (45) and (46) are already known for the case $r=2$, (cfr. [5], formulas (317) resp. (318), p. 58).

It is easily proved that the sums in the right members of equations (45) and (46) represent algebraic numbers. Therefore quantities $K_r(2m+1)$, $r=2, 3, 4, \ldots$, $m=0, 1, 2, \ldots$, and $L_r(2m)$, $r=2, 3, 4, \ldots$, $m=1, 2, 3, \ldots$, are not algebraically independent in the sense that there exists function $F(x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j)$, $i, j \geqq 1$, constructed by algebraic numbers and algebraic operations only, such that if substitutions $x_s = K_{q_s}(2m_s+1)$, $s=1, 2, \ldots, i$, $y_t = L_{r_t}(2n_t)$, $t=1, 2, \ldots, j$ are effected, the equation

$$F(x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j) = 0$$

is satisfied.

Now, completely fruitless attempts to obtain something different from trivial identity by passing to limit when $r \to \infty$ in formulas (45) and (46) (and some other reasons) are conductive to the following

Hypothesis. The numbers $K(2m+1)$ and $L(2n)$, where for m and n may be taken any set of positive integers (which contains at least two elements), are algebraically independent in the above sense.

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