A note on bi-ideals and quasi-ideals in semigroups

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The purpose of this paper is twofold: In the first section we will investigate the equivalence relation $\mathscr{B}$ [3] which is finer than $\mathscr{H}$. We will show (1. 8) that $\mathscr{B}$ relates exactly those elements which generate the same bi-ideal, and that (1. 10) the intersection of the principal bi-ideal generated by a given element with its $\mathscr{H}$-class is its $\mathscr{B}$-class. We will also show (1. 11) that if a $\mathscr{B}$-class contains an irregular element, a, it is just $\{a\}$, otherwise it is an $\mathscr{H}$-class.

In the second section we will show (2. 9) that a bi-ideal which is generated by a regular element is equal to the quasi-ideal generated by that element, proving (2. 10) that 0-minimal bi-ideals which are groups union $\{0\}$ are 0-minimal quasi-ideals.

The notation of CLIFFORD and PRESTON [2] will be used.

1. $\mathscr{B}$-*Class Structure*

(1. 1) *Definition.* A (non-empty) subset B of a semigroup S is a *bi-ideal* if $B^2 \cup BSB = BS^1B \subseteq B$.

The following equivalence relation was given by KAPP [3].

(1. 2) *Definition.* For $a, b \in S$, a given semigroup, we write $a\mathscr{B}b$ if 1) $a=b$ or 2) there exists $u, v \in S$ such that $aua=b$ and $bvb=a$. As usual, B_a denotes the $\mathscr{B}$-class of a, and $a\bar{\mathscr{B}}b$ will be used when a and b are not $\mathscr{B}$-related.

(1. 3) *Proposition.* The relation $\mathscr{B}$ defined in (1. 2) is an equivalence relation, indeed $\mathscr{B} \subseteq \mathscr{H}$. [[3] Proposition (1. 3)].

The following provides a simple example of a semigroup in which $\mathscr{B} \subset \mathscr{H}$ ($\subset$ is used for proper containment).

(1. 4) *Example.* Let $S=(Z/(8), \cdot)$. Then $\bar{2}\mathscr{H}\bar{6}$ but $\bar{2}\bar{\mathscr{B}}\bar{6}$ since for every $u \in S$, $\bar{2}u\bar{2}=\bar{4}u$ which is always $\bar{4}$ or $\bar{0}$.

(1. 5) *Proposition.* If S is a semigroup and $x \in S$ is regular, then $B_x = H_x$.

PROOF. From (1. 3) we have $B_x \subseteq H_x$ for any x. If x is a regular, then every element of D_x and hence H_x is regular [[2] Theorem 2. 11 (i)]. Let $u \in H_x$, if $u=x$, $u \in B_x$. Now assume $u \neq x$, then since u and x are regular, there exists $t_1, t_2 \in S$ such that $u=ut_1u$ and $x=xt_2x$. Since $x\mathscr{H}u$, and $x \neq u$, there are $s_1, s_2, s_3, s_4 \in S$ such that $x=s_1u=us_2$ and $u=s_3x=xs_4$. Thus $x=s_1u=(s_1u)t_1u=(us_2)t_1u=u(s_2t_1)u$, and $u=s_3x=(s_3x)t_2x=(xs_4)t_2x=x(s_4t_2)x$. Thus in either case, $u \in B_x$ and thus $H_x \subseteq B_x$, whence $B_x = H_x$.

(1. 6) *Definition.* Let S be a semigroup and $x \in S$, then the *principal bi-ideal*, $B(x)$, *generated by* x is the smallest bi-ideal of S containing x. Clearly $B(x) = x \cup xS^1 x$.

K. KAPP has shown:

(1. 7) *Proposition.* If A is a bi-ideal in a semigroup, then $A = \bigcup_{a \in A} B_a$, *i.e.*, any bi-ideal is the union of its $\mathscr{B}$-classes. [[3] Proposition (1. 4)].

One easily checks the following correspondence between the $\mathscr{B}$-relation and the bi-ideals of a semigroup.

(1. 8) *Proposition.* Let S be a semigroup. Then for $x, y \in S$, $x \mathscr{B} y$ if and only if $B(x) = B(y)$.

Now we investigate the relation between principal bi-ideals and $\mathscr{H}$-classes.

(1. 9) **Theorem.** *Let S be a semigroup and $x \in S$. If $|B(x) \cap H_x| > 1$, then x is a regular element of S.*

PROOF. If $|B(x) \cap H_x| > 1$, there exists $u \in (B(x) \cap H_x) \setminus \{x\}$. Thus $u \in B(x) \setminus \{x\} = xS^1 x \setminus \{x\}$. It follows that either $u = x^2$ or that there is an $s \in S$ with $u = xsx$. If $x^2 = u \in H_x$, then H_x is a group [[2] Theorem 2. 16] and x is clearly regular. On the other hand, if $u = xsx$, then $u \in H_x \setminus \{x\}$, implies that there exist $s_1, s_2 \in S$ such that $x = us_1 = s_2 u$. Thus $x = us_1 = xsxs_1$ and $x = s_2 u = s_2 xsx$. But then $xs_1 = s_2(xsx)s_1 = s_2 x$ and $x = xsxs_1 = xss_2 x$. Whence x is regular.

(1. 10) *Corollary.* $B(x) \cap H_x = B_x$.

PROOF. Since $\mathscr{B} \subseteq \mathscr{H}$, it follows from (1. 8) that $B_x \subseteq H_x \cap B(x)$.

If $|B(x) \cap H_x| = 1$, clearly $B_x = B(x) \cap H_x = \{x\}$. If $|B(x) \cap H_x| > 1$, then x is regular, and by (1. 5), $B_x = H_x \supseteq B(x) \cap H_x$ so that $B_x = B(x) \cap H_x$ and the result follows in either case.

(1. 11) *Corollary.* Let S be a semigroup and $a \in S$. Then either i) a is irregular and $B_a = \{a\}$, or ii) a is regular and $B_a = H_a$.

2. 0-*Minimal Bi-ideals*

We now consider the relationship between 0-minimal bi-ideals and 0-minimal quasi-ideals.

(2. 1) *Definition.* A non-zero bi-ideal B of a semigroup S with 0 is said to be 0-*minimal* if there is no bi-ideal B' of S with $\{0\} \subset B' \subset B$.

(2. 2) *Definition.* A (non-empty) subset Q of a semigroup S is called *a quasi-ideal* if $QS \cap SQ \subseteq Q$. Let $x \in S$, then the *principal quasi-ideal generated*, $Q(x)$, *by* x is the smallest quasi-ideal of S containing x. Clearly $Q(x) = xS^1 \cap S^1 x$.

(2. 3) *Definition.* A non-zero quasi-ideal Q of a semigroup S with 0 is 0-*minimal* if there is no quasi-ideal Q' of S with $\{0\} \subset Q' \subset Q$.

We recall:

(2. 4) *Proposition.* Every quasi-ideal of a semigroup is a bi-ideal. In a regular semigroup, every bi-ideal is also a quasi-ideal. [[2] p. 85, ex. 18.]

(2. 5) **Theorem.** *Let S be a semigroup with* 0. *A* 0*-minimal bi-ideal B of S is either null or a group union* $\{0\}$ [[3] Theorem (1. 8)] and

(2. 6) **Theorem.** *Let S be a semigroup with* 0. *If B is a* 0*-minimal bi-ideal which is a group union* $\{0\}$ *and can be written as the product of a* 0*-minimal right ideal and a* 0*-minimal left ideal, then B is also a* 0*-minimal quasi-ideal.* [[3] Theorem (2. 15).]

K. KAPP [3] went on to ask whether or not any 0-minimal bi-ideal which was a group union $\{0\}$ was a quasi-ideal, since not every such bi-ideal can be written as a product of a 0-minimal right ideal and a 0-minimal left ideal. The following theorem and corollaries answer this question.

(2. 7) **Theorem.** *Let S be a semigroup with* 0. *A bi-ideal is* 0*-minimal if and only if it is a non-zero $\mathscr{B}$-class union* $\{0\}$. [[3] Theorem (1. 7).]

Clearly

(2. 8) **Lemma.** *If B is a* 0*-minimal bi-ideal, then for $x\in B\setminus\{0\}$, $B=B(x)$.*

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(2. 9) **Theorem.** *If a is a regular element of a semigroup S, then $B(a)=Q(a)$.*

PROOF. Clearly $B(a)\subseteq Q(a)$. Let $x\in Q(a) = aS^1\cap S^1a$, then there are $s_1, s_2\in S^1$ such that $x=as_1=s_2a$. But a is regular, therefore $a=asa$ for some $s\in S$. Combining the above equations we have $x=as_1=asas_1=ass_2a\in aS^1a$, so that $Q(a)\subseteq B(a)$. The result follows.

(2. 10) *Corollary.* If B is a 0-minimal bi-ideal of S, a semigroup with zero, and $a\in B\setminus\{0\}$ is regular, then B is a 0-minimal quasi-ideal of S.

(2. 11) *Corollary.* If B is a 0-minimal bi-ideal which is a group union zero, then B is a 0-minimal quasi-ideal.

Finally:

(2. 12) *Corollary.* If B is a 0-minimal bi-ideal of S and $|B|>2$, then B is a 0-minimal quasi-ideal of S.

PROOF. By (2. 8), $B=B(x)$ where $x\in B\setminus\{0\}$. But by (2. 7), we have $B\setminus\{0\}=B_x$, hence $|B(x)\cap H_x| = |B_x|>1$ and x is regular by (1. 9). Thus by (2. 9), B is a 0-minimal quasi-ideal.

Bibliography

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