

On functions satisfying $\operatorname{Re} \left[\frac{f(z)}{z} \right] > \alpha$

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Let S_α denote the class of functions

$$(1) \quad f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$$

which are regular analytic in the unit disc $|z| < 1$ and satisfy the condition

$$(2) \quad \operatorname{Re} \left[\frac{f(z)}{z} \right] > \alpha \quad (0 \leq \alpha < 1) \quad \text{for } |z| < 1.$$

In a recent paper [1] K. YAMAGUCHI has proved that if $f(z) \in S_0$, then

$$(3) \quad \operatorname{Re} [f'(z)] \geq \frac{1-2r-r^2}{(1+r)^2} \quad \text{for } 0 \leq r < \sqrt{2}-1, \quad z = re^{i\theta}.$$

In the present paper, we increase the constant $\sqrt{2}-1$ to $\frac{1}{2}$. We shall however, consider a class T_α of functions belonging to the class S_α with the additional condition that the second coefficient a_2 is fixed. Without loss of generality we may assume $a_2 \geq 0$ since $e^{-i\theta}f(ze^{i\theta})$ belongs to T_α whenever $f(z)$ belongs to T_α .

Let P denote the class of regular, analytic functions in the unit disc $|z| < 1$ whose real part is positive and which take the value 1 at the origin.

We need the following

Lemma. If $p(z) \in P$ and $|p'(0)| = 2b > 0$ then

$$(4) \quad \operatorname{Re} [p(z) + zp'(z)] \geq \frac{1-4r^2-4br^3-r^4}{(1+2br+r^2)^2}, \quad |z| = r,$$

for $0 \leq |z| < r_0$ where r_0 is the smallest positive root of

$$(5) \quad 1-3r^2-2br^3 = 0$$

The result is skarp.

PROOF. Since $p(z) \in P$, we can express it in the form

$$(6) \quad p(z) = \frac{1-\omega(z)}{1+\omega(z)}$$

where $\omega(z)$ is regular analytic in $|z| < 1$ and satisfies the conditions $\omega(0) = 0$, $|\omega'(0)| = b$ and $|\omega(z)| < 1$.

Using (6) we obtain

$$(7) \quad p(z) + zp'(z) = \frac{1 - \omega(z)}{1 + \omega(z)} - \frac{2z\omega'(z)}{(1 + \omega(z))^2}.$$

Such functions $\omega(z)$ are known [3, p. 167] to satisfy the inequalities

$$(8) \quad \frac{r(b-r)}{1-br} \leq |\omega(z)| \leq \frac{r(b+r)}{1+br}.$$

Let

$$(9) \quad \varphi(z) = \frac{\omega(z)}{z}$$

then $\varphi(z)$ is analytic in $|z| < 1$ and satisfies the inequality [2, p. 18]

$$(10) \quad |\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \quad \text{for } |z| < 1.$$

From (9) and (10) we get

$$(11) \quad |z\omega'(z) - \omega(z)| \leq \frac{r^2 - |\omega(z)|^2}{1 - r^2}, \quad |z| = r.$$

(7) gives in conjunction with (11)

$$(12) \quad \operatorname{Re}[p(z) + zp'(z)] \leq -1 + 2 \operatorname{Re} \left[\frac{1}{(1 + \omega(z))^2} \right] - \frac{2(r^2 - |\omega(z)|^2)}{(1 - r^2)|1 + \omega(z)|^2},$$

which on substituting

$$(13) \quad \omega_1(z) = \frac{1}{1 + \omega(z)}$$

reduces to

$$(14) \quad \operatorname{Re}[p(z) + zp'(z)] \leq -1 + 2 \operatorname{Re} \omega_1^2 - 2 \frac{(r^2 |\omega_1|^2 - |1 - \omega_1|^2)}{(1 - r^2)}.$$

In view of the second inequality in (8) it is easy to see that

$$(15) \quad \left| \omega_1(z) - \frac{1}{1 - r^2} \right| \leq \frac{r}{1 - r^2}.$$

However, because of the first inequality in (8) all values in the interior of the circle given by the (15) are not taken.

Putting $\omega_1(z) = u + iv$ and then denoting the right hand side of (14) by $S(u, v)$ we get

$$(16) \quad S(u, v) = \frac{1 + r^2}{1 - r^2} - \frac{4u}{1 - r^2} + 4u^2,$$

which shows that $S(u, v)$ is a function of u only, say $M(u)$. The absolute minimum of $M(u)$ in $(0, \infty)$ is attained at $u_0 = \frac{1}{2(1 - r^2)}$ and equals

$$(17) \quad M(u_0) = \frac{-r^4}{(1 - r^2)^2}.$$

Since $\omega(z)$ is real whenever $\omega_1(z)$ is real, we have

$$u = \frac{1}{1 + \operatorname{Re} \omega(z)} \cong \frac{1 + br}{1 - r^2}$$

and

$$u = \frac{1}{1 + \operatorname{Re} \omega(z)} \cong \frac{1 + br}{1 + 2br + r^2},$$

where we have used the inequalities

$$\frac{-r(b+r)}{1+br} \leq \operatorname{Re} \omega(z) \leq \frac{r(b+r)}{1+br}.$$

It is easy to see that $u_0 < \frac{1+br}{1-r^2}$ but u_0 is not always greater than $\frac{1+br}{1+2br+r^2}$.

In the latter case the minimum of $M(u)$ in the segment $\left[\frac{1+br}{1+2br+r^2}, \frac{1+br}{1-r^2} \right]$ is attained at $u = \frac{1+br}{1+2br+r^2}$ and equals

$$(18) \quad M\left(\frac{1+br}{1+2br+r^2}\right) = \frac{1-4r^2-4br^3-r^4}{(1+2br+r^2)^2}.$$

(18) will hold for those values of r for which

$$\frac{1}{2(1-r^2)} \cong \frac{1+br}{1+2br+r^2}$$

that is,

$$(19) \quad 1 - 3r^2 - 2br^3 \geq 0.$$

Since (19) holds at $r=0$ but it does not hold at $r=1$. Hence the minimum value (18) is acceptable for $0 \leq r < r_0$ where r_0 is the smallest positive root of $1 - 3r^2 - 2br^3 = 0$.

Inequality (4) follows from (18).

The equality sign in (4) is attained for the function

$$(20) \quad P(z) = \frac{1-z^2}{1+2bz+z^2} \quad \text{at} \quad z = r.$$

This completes the proof of the lemma.

Remark. From (17) we find that

$$\operatorname{Re}[p(z) + zp'(z)] \geq \frac{-r^4}{(1-r^2)^2} \quad \text{for} \quad r_0 \leq r < 1,$$

but the bound is not sharp.

Theorem 1. If $f(z) \in T_\alpha$, then

$$(21) \quad \operatorname{Re} [f'(z)] \cong \frac{(2\alpha-1)r^4 + 4b(2\alpha-1)r^3 + (-4+6\alpha+4\alpha b^2)r^2 + 4\alpha br + 1}{(1+2br+r^2)^2}$$

for $0 \leq r < r_0$, $|z|=r$, $a_2=2b$ and r_0 is the smallest positive root of $1-3r^2-2br^3=0$. The bound is sharp.

PROOF. Since $f(z) \in T_\alpha$, $\operatorname{Re} \left[\frac{f(z)}{z} \right] > \alpha$ and hence

$$(22) \quad \frac{f(z)/z - \alpha}{1 - \alpha} = p(z)$$

where $p(z) \in P$. Differentiating (22) we obtain

$$f'(z) = \alpha + (1 - \alpha)[p(z) + zp'(z)].$$

The theorem now follows easily on using the above lemma. The extremal function is given by

$$f(z) = \frac{z(1 + 2b\alpha z + (2\alpha - 1)z^2)}{1 + 2bz + z^2}$$

Corollary 1.1. If $f(z) \in S_\alpha$ then on putting $b=1$ in (21) we have

$$(23) \quad \operatorname{Re} [f'(z)] \cong \frac{(2\alpha-1)r^2 + 2(2\alpha-1)r + 1}{(1+r)^2} \quad \text{for } 0 \leq r < \frac{1}{2}.$$

The bound is sharp.

On putting $\alpha=0$ in (23) we get

$$(24) \quad \operatorname{Re} f'(z) \cong \frac{1-2r-r^2}{(1+r)^2} \quad \text{for } 0 \leq r < \frac{1}{2}.$$

Corollary 1.2. On putting $b=0$ in (21) we obtain

$$(25) \quad \operatorname{Re} f'(z) \cong \frac{(2\alpha-1)r^4 + (6\alpha-4)r^2 + 1}{(1+r^2)^2} \quad \text{for } 0 \leq r < \frac{1}{\sqrt{3}}.$$

This is the sharp estimate for odd functions $f(z) = z + a_3 z^3 + \dots$ such that $\operatorname{Re} \left[\frac{f(z)}{z} \right] > \alpha$ since an odd function has vanishing even coefficients and the extremal function in this case reduces to $f(z) = z(1 + (2\alpha-1)z^2)/(1+z^2)$.

We shall now consider some applications of Theorem 1.

Theorem 2. If $f(z) \in T_\alpha$, then

$$(26) \quad f(z) \text{ is univalent in } |z| < r_0 \text{ for } \frac{1}{2} \leq \alpha < 1;$$

and

$$(27) \quad f(z) \text{ is univalent in } |z| < \min(r_0, r_1) \text{ for } 0 \leq \alpha < \frac{1}{2},$$

where r_0 and r_1 are the smallest positive roots of $1 - 3r^2 - 2br^3 = 0$ and $(2\alpha - 1)r^4 + 4b(2\alpha - 1)r^3 + (-4 + 6\alpha + 4ab^2)r^2 + 4abr + 1 = 0$ respectively.

PROOF. It is easily seen that for $\frac{1}{2} \leq \alpha < 1$, right hand side of (21) is always positive and consequently $\operatorname{Re} f'(z) < 0$ for $|z| < r_0$. For $0 \leq \alpha < \frac{1}{2}$, $\operatorname{Re} f'(z) > 0$ provided

$$K(r) = (2\alpha - 1)r^4 + 4b(2\alpha - 1)r^3 + (-4 + 6\alpha + 4ab^2)r^2 + 4abr + 1 > 0.$$

Let r_1 denote the smallest positive root of $K(r) = 0$. Then $\operatorname{Re} f'(z) > 0$ for $|z| < \min(r_0, r_1)$.

Theorem 2. now follows immediately from the following well known *Wolff—Nos- hiro Lemma*: If $f(z)$ is analytic in $|z| < R$ and $\operatorname{Re} f'(z) > 0$ in $|z| < R$, then $f(z)$ is univalent in $|z| < R$.

Corollary 2.1. If $f(z) \in S_\alpha$, then $f(z)$ is univalent in $|z| < \sqrt{\frac{2-2\alpha}{1-2\alpha}} - 1$ for $0 \leq \alpha \leq \frac{1}{10}$.

The result follows by combining Theorem 2 with Corollary 1.1.

Theorem 3. If $f(z) = z + a_2 z^2 + \dots + a_n z^n + \dots$ belongs to S_α , then the partial sums

$$S_n(z) = z + a_2 z^2 + \dots + a_n z^n \quad (n=2, 3, \dots)$$

are univalent in $|z| < \frac{1}{4(1-\alpha)}$ for $0 \leq \alpha \leq \frac{1}{10}$.

PROOF. The function

$$(28) \quad p(z) = \frac{f(z)/z - \alpha}{1 - \alpha} = 1 + \frac{1}{1 - \alpha} (a_2 z + a_3 z^2 + \dots)$$

belongs to P . Hence, by Caratheodory — Toeplitz's theorem

$$(29) \quad |a_n| \leq 2(1 - \alpha) \quad (n=2, 3, \dots).$$

Now

$$\operatorname{Re} [S'_n(z)] = \operatorname{Re} f'(z) - \operatorname{Re} \left[\sum_{m=n+1}^{\infty} m a_m z^{m-1} \right],$$

which gives together with (23) and (29)

$$\operatorname{Re} [S'_n(z)] \geq \frac{(2\alpha - 1)r^2 + 2(2\alpha - 1)r + 1}{(1 + r)^2} - \frac{2(1 - \alpha)(n + 1 - nr)r^n}{(1 - r)^2}, \quad |z| = r.$$

If $r = \frac{1}{4(1-\alpha)} \leq \left(\sqrt{\frac{2-2\alpha}{1-2\alpha}} - 1 \right)$, then

$$(n + 1 - nr)r^n \leq (5 - 4r)r^4 \quad (n=4, 5, \dots),$$

and hence for $r = \frac{1}{4(1-\alpha)}$,

$$\operatorname{Re} S'_n(z) \cong \frac{7-6\alpha}{(5-4\alpha)^2} - \frac{(4-5\alpha)}{8(1-\alpha)^2(3-4\alpha)^2} > 0 \quad \text{for } 0 \leq \alpha \leq \frac{1}{10} \quad \text{and } n \geq 4.$$

The real part of $S'_n(z)$ is harmonic function and therefore by the principle of minimum modulus we have

$$\operatorname{Re} S'_n(z) > 0 \quad \text{for } |z| \leq \frac{1}{4(1-\alpha)}, \quad n \geq 4, \quad 0 \leq \alpha \leq \frac{1}{10}.$$

The theorem is thus proved for $n \geq 4$.

Let us now consider the case $n=2$.

We have by (29)

$$\operatorname{Re} S'_2(z) = \operatorname{Re} (1 + 2a_2 z) \geq 1 - 4(1-\alpha)|z| > 0 \quad \text{for } |z| < \frac{1}{4(1-\alpha)}.$$

Finally we consider the case $n=3$.

Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ belong to P , then by Caratheodory—Toeplitz's theorem we have

$$(30) \quad |2p_2 - p_1^2| \leq 4 - |p_1|^2.$$

Let

$$c = 2p_2 - p_1^2, \quad p_1 z = \alpha' + i\beta', \quad \sqrt{c} z = \gamma' + i\delta', \quad |z| \leq \frac{1}{4(1-\alpha)}.$$

Then on substituting the power series expansion for $p(z)$ in (28) we obtain

$$S_3(z) = z + a_2 z^2 + a_3 z^3 = z + p_1(1-\alpha)z^2 + p_2(1-\alpha)z^3.$$

Therefore,

$$\begin{aligned} S'_3(z) &= 1 + (1-\alpha)(2p_1 z + 3p_2 z^2) = 1 + (1-\alpha) \left[2p_1 z + \frac{3}{2}(c + p_1^2)z^2 \right]. \\ \operatorname{Re} S'_3(z) &= 1 + (1-\alpha) \left[2\alpha' + \frac{3}{2}(\gamma'^2 - \delta'^2) + \frac{3}{2}(\alpha'^2 - \beta'^2) \right] = \\ &= 1 + (1-\alpha) \left[2\alpha' + \frac{3}{2}\alpha'^2 - \frac{3}{2}\beta'^2 + \frac{3}{2}\gamma'^2 - \frac{3}{2}(|cz^2| - \gamma'^2) \right] \cong \\ &\cong 1 + (1-\alpha) \left[2\alpha' + \frac{3}{2}\alpha'^2 - \frac{3}{2}\beta'^2 + \frac{3}{2}\gamma'^2 - \frac{3}{2} \left(\frac{4 - |p_1|^2}{16(1-\alpha)^2} - \gamma'^2 \right) \right] \cong \\ &\cong 1 + (1-\alpha) \left[2\alpha' + \frac{3}{2}\alpha'^2 - \frac{3}{2}\beta'^2 + \frac{3}{2}\gamma'^2 - \frac{3}{2} \left(\frac{1}{4(1-\alpha)^2} - \alpha'^2 - \beta'^2 - \gamma'^2 \right) \right] = \\ &= 1 + (1-\alpha) \left[2\alpha' + 3\alpha'^2 + 3\gamma'^2 - \frac{3}{8(1-\alpha)^2} \right] \cong 1 - \frac{3}{8(1-\alpha)} + (1-\alpha)[2\alpha' + 3\alpha'^2] \cong \\ &\cong \frac{7-8\alpha-8\alpha^2}{24(1-\alpha)} > 0 \quad \text{for } 0 \leq \alpha \leq \frac{1}{10}. \end{aligned}$$

To show that the result is sharp, consider the function

$$f(z) = z \left[\frac{1 + (1 - 2\alpha)z}{1 - z} \right] = z + 2(1 - \alpha)z^2 + \dots$$

which belongs to S_α . For this function

$$S_2(z) = z + 2(1 - \alpha)z^2,$$

and $S_2'(z) = 0$ when $z = -\frac{1}{4(1 - \alpha)}$.

This completes the proof of the theorem.

References

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