

Local growth of the number of prime divisors of consecutive integers

By I. KÁTAI (Budapest)

1. Let $\omega(n)$ denote the number of the different prime factors of n . Similarly, let $\Omega(n)$ be the number of all the prime factors of n each counted by its multiplicity. Let $d(n)$ be the number of divisors of n .

As it is well known, $\omega(n) = (1 + o(1)) \cdot \log \log n$ holds for all n except a set of zero density.

We shall investigate now the maximal term of $\omega(n+1), \dots, \omega(n+k)$.

Let

$$(1.1) \quad O_k(n) = \max_{j=1, \dots, k} \omega(n+j),$$

$$(1.2) \quad o_k(n) = \min_{j=1, \dots, k} \omega(n+j).$$

Let $\psi(z) = z \log \frac{z}{e} + 1$ be defined for $z \geq 1$, and $\varrho(u)$ denote its inverse function ($u \geq 0$).

I guess that the inequality

$$(1.3) \quad 1 - \varepsilon \leq \frac{O_k(n)}{\varrho\left(\frac{\log k}{\log \log n}\right) \log \log n} \leq 1 + \varepsilon$$

holds for almost all n , uniformly in $k (= 1, 2, \dots)$, ε being an arbitrary positive constant.

At this time, I can prove only the right hand side of (1.3), i.e.

Theorem 1. *For every constant $\varepsilon < 0$ and for all n except a set of zero density the inequalities*

$$(1.4) \quad O_k(n) \leq (1 + \varepsilon) \varrho\left(\frac{\log k}{\log \log n}\right) \log \log n,$$

$$(1.5) \quad O_k(n) \leq \left\{ \varrho\left(\frac{\log k}{\log \log n}\right) + \varepsilon \right\} \log \log n$$

hold uniformly in k .

We shall prove only (1.4), the proof of (1.5) is similar. We use the notation $x_1 = \log x$, $x_2 = \log x_1$, $x_3 = \log x_2$. The letters A, B, C denote suitable positive constants not necessarily the same at every occurrence; p will denote prime numbers.

Let

$$D(x, z) = \sum_{n \leq x} z^{\omega(n)} \quad (z \geq 1).$$

We prove the inequality

$$(1.6) \quad D(x, z) \leq B[(z-1)x_2]^A x_1^{z-1} \quad (x > 10)$$

Assume that x is large enough. Let A be such a large integer that

$$\sum_{p \leq x^{1/A}} \frac{1}{p} \leq x_2$$

holds. The symbol Q_r denotes a general square free integer having exactly r prime factors. Since for $Q_r \leq x$, Q_r has at most A prime factors greater than $x^{1/A}$ we have

$$\begin{aligned} \sum_{Q_r \leq x} \frac{1}{Q_r} &\leq \frac{1}{(r-A)!} \left(\sum_{p \leq x^{1/A}} \frac{1}{p} \right)^{r-A} \left(\sum_{p \leq x} \frac{1}{p} \right)^A < \\ &< \frac{1}{(r-A)!} (x_2 + B)^A x_1^{r-A} < \left(1 + \frac{B}{x_2} \right)^A \frac{1}{(r-A)!} x_2^r. \end{aligned}$$

Using the relation

$$z^{\omega(n)} = \sum_{r=0}^{\infty} \sum_{Q_r | n} w^r, \quad w = z-1,$$

we have

$$D(x, z) = \sum_{r=0}^{\infty} \sum_{Q_r \leq x} w^r \left[\frac{x}{Q_r} \right] \leq x \sum_{r=0}^{\infty} w^r \sum_{Q_r \leq x} \frac{1}{Q_r}.$$

So by (1.7) we obtain

$$D(x, z) < Bx x_2^A w^A \sum_{r=A}^{\infty} \frac{w^{r-A} x_2^{r-A}}{(r-A)!} = Bx x_2^A w^A x_1^w,$$

which was desired.

Let $M(x, z)$ be the number of those $n, \leq x$ for which $\omega(n) > zx_2$. Let $\eta > 0$ be an arbitrary constant. By (1.6) we have

$$M(x, z(1+\eta)) z^{z(1+\eta)x_2} \leq Bx[(z-1)x_2]^A x_1^{z-1},$$

whence

$$(1.8) \quad M(x, z(1+\eta)) \leq Bx[(z-1)x_2]^A \cdot x_1^{z-1-z(1+\eta)\log z(1+\eta)}.$$

Instead of (1.4) we prove that

$$(1.9) \quad O_k(n) \leq (1+\varepsilon) \varrho \left(\frac{\log k}{x_2} \right) x_2$$

for all $n, \leq x$ except at most $o(x)$ of n 's.

Let $z = 1 + \varepsilon'$, $\eta = \varepsilon'$, ε' be a small positive constant. By (1.8) we have

$$M(x, (1 + \varepsilon')^2) \leq Bx[\varepsilon' x_2]^A x_1^{\varepsilon' - (1 + \varepsilon')^2 \log(1 + \varepsilon')^2} < Bx x_1^{-\varepsilon'/2}.$$

Let k_0 be such an integer that

$$\frac{\log k_0}{x_2} < \frac{\varepsilon'}{\varepsilon}.$$

Then

$$(1.10) \quad k_0 M(x, (1 + \varepsilon')^2) < Bx \cdot x_1^{-\varepsilon'/\varepsilon}.$$

Hence it follows that $\max_{k \leq k_0} O_k(n) \leq (1 + \varepsilon')^2 x_2$ except $o(x)$ of n 's. Choosing $(1 + \varepsilon')^2 < (1 + \varepsilon)$, we obtain that (1.9) holds, if $\frac{\log k_0}{x_2} \leq \frac{\varepsilon'}{3}$ ($\varepsilon' > 0$).

Let now $(\log k)/x_2 \geq \varepsilon'/3$, $k = x_1^{\psi(z)}$, i.e. $z = \varrho\left(\frac{\log k}{x_2}\right)$. Observing that

$$[x_2(z-1)]^A < x_1^{(\eta/2)z \log z} \text{ for all large } x, z > 1 + \frac{\varepsilon'}{3},$$

we have

$$(1.11) \quad kM(x, z(1 + \eta)) < Bx \cdot x_1^{-(\eta/2)z \log z}$$

Let

$$k_m = [m^{4/\eta}] + 1 \quad z_m = \varrho\left(\frac{\log k_m}{x_2}\right), \quad \psi(z_m) = \frac{\log k_m}{x_2}.$$

Since

$$z \log z = \psi(z) + z - 1 \geq \psi(z) \quad (z \geq 1),$$

from (1.11) we deduce

$$(1.12) \quad k_m M(x, z_m(1 + \eta)) < Bx k_m^{-\eta/2} < Bx \cdot m^{-2}.$$

Hence

$$\sum_{\log k_m \leq (\varepsilon'/3)x_2} k_m M(x, z_m(1 + \eta)) = o(x).$$

Taking $\eta = \varepsilon$ we obtain that (1.9) holds for all, $k = k_m$.

Let now $k_m < k < k_{m+1}$. It is obvious that $O_k(n) \leq O_{k_m}(n)$. To finish the proof it is enough to show that

$$\varrho\left(\frac{\log k_{m+1}}{x_2}\right) \leq (1 + \eta) \varrho\left(\frac{\log k_m}{x_2}\right),$$

if m is large enough; η being an arbitrary positive constant. But this is an obvious consequence of the fact that $\varrho(u)$ has a bounded derivative in $u \geq u_0 (> 0)$.

2. In a similar way we can prove the following assertion. Let $\psi(z) = z \log z - z + 1$ be defined for $0 < z \leq 1$, and $\bar{\varrho}(u)$ its inverse function.

Theorem 2. For every constant $\varepsilon > 0$ and almost all n , the inequality

$$(2.1) \quad o_k(n) \cong (1 - \varepsilon) \bar{q} \left(\frac{\log k}{\log \log n} \right) \log \log n$$

holds uniformly in k .

This is an easy consequence of the fact that the number of $n \leq x$ having exactly r different prime factors is

$$(1 + o(1)) \frac{x}{x_1} \frac{x_2^{r-1}}{(r-1)!}.$$

I think that

$$o_k(n) \cong (1 + \varepsilon) \bar{q} \left(\frac{\log k}{\log \log n} \right) \log \log n$$

holds for almost all n and every k .

3. Finally we sketch the proof of the following

Theorem 3. Let $k = x_1^{2 \log 2 - 1} \cdot e^{h(x) \sqrt{x_2}}$, where $h(x) \rightarrow \infty$, $h(x) = o(\sqrt{x_2})$. Then

$$(3.1) \quad O_k(n) = 2(1 + o(1))x_2$$

for all $n \leq x$ except $o(x)$ of them.

We need only to prove that

$$(3.2) \quad O_k(n) \cong 2(1 - \varepsilon)x_2.$$

To prove this we use a result due to P. ERDŐS [1]. Let

$$(3.3) \quad A(n) = \sum_{i=1}^k d(n+i),$$

Σ_1 denotes that we sum over those integers for which $\Omega(n+i) \leq 2x_2 + \frac{1}{2}h(x)\sqrt{x_2}$.

Then

$$(3.4) \quad \sum_{n \leq x} (A(n) - kx_1)^2 = o(xk^2x_1^2).$$

We split the sum $A(n)$ into two parts

$$A(n) = A_1(n) + A_2(n),$$

where in

$$A_1(n) \quad \Omega(n) - \omega(n) \leq \eta x_2, \quad \text{and in} \quad A_2(n) \quad \Omega(n) - \omega(n) > \eta x_2.$$

We can prove easily that

$$\sum_{n \leq x} A_2^2(n) = o(xk^2x_1^2),$$

whence by (3.4)

$$(3.5) \quad \sum_{n \leq x} (A_1(n) - kx_1)^2 = o(xk^2x_1^2)$$

follows.

Let now

$$B(n) = \sum_{i=1}^k d(n+i), \quad \omega(n+i) < (2-\varepsilon)x_2.$$

Similarly, we can prove easily that

$$\sum_{n \leq x} B^2(n) = o(xk^2x_1^2).$$

Hence by (3.5) we have

$$A_1(n) - B(n) = (1 + o(1))kx_1 > 0$$

for almost all n .

Observing that

$$A_1(n) - B(n) = \sum_{i=1}^k d(n+i), \quad \omega(n+i) > (2-\varepsilon)x_2$$

our theorem follows immediately.

References

- [1] P. ERDŐS, Asymptotische Untersuchungen über die Anzahl der Teiler von n , *Math. Ann.* **169** (1967), 230—238.

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