

A characterization of biperfect topogenous orders

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As is known, biperfect topogenous orders play an important role in the syn-topogenous build-up of general topology (see [1], in particular Chapter 5). Their importance is largely due to the fact that a natural one-to-one correspondence exists between the biperfect topogenous orders and the reflexive relations defined on a given set ([1], theorem (5. 39)—(5. 41)).

In this short note we are going to establish a result, running parallel as it were, to the theorem just quoted. Bipерfect topogenous orders over a given set will be shown to be equivalent to a class of entities, more complicated than reflexive relations, but endowed with a simple meaning that may appeal to our intuition.

We start with two definitions, one of them well-known:

Definition 1. A biperfect topogenous order on a set E is a relation $<$ defined on the set of all subsets of E , satisfying the following axioms:

- (O1) $\emptyset < \emptyset$, $E < E$;
- (O2) $A < B$ implies $A \subseteq B$;
- (O3) $A \subseteq A' < B' \subseteq B$ implies $A < B$;
- $(B_{\cap})$ $A < B_i$ for $i \in I$ implies $A < \bigcap \{B_i | i \in I\}$;
- $(B_{\cup})$ $A_i < B$ for $i \in I$ implies $\bigcup \{A_i | i \in I\} < B$. ■

Definition 2. A kernelled covering of a set E is a triple (γ, π, f) , with γ a covering of E , π a partition of E and f a one-to-one mapping of π onto γ , satisfying $P \subseteq f(P)$ for any $P \in \pi$. ■

The intuitive background of the second definition is quite simple. If γ is a covering of E , then for each $x \in E$ let us choose a set $G \in \gamma$ containing x . (In other words: for each $x \in E$, choose a member of $\text{St}(x, \gamma)$.) Now, two points of E will be considered equivalent, if the same set has been chosen for them. If we still make correspond to each equivalence class its chosen covering set, then we obtain a kernelled covering (γ_1, π, f) , γ_1 being the class of chosen sets¹⁾, π the partition formed by the equivalence classes just defined, with the correspondence f clearly one-to-one.

Thus we see that each covering of a set gives raise in a natural way to kernelled coverings, these being essentially the same as the selection functions operating on the stars of the space's points.

¹⁾ γ_1 is a subset of γ . There are cases when this subset is necessarily proper. (E.g. cover a set by all its subsets.)

Let now $<$ be a biperfect topogenous order. Put

$$U_<(x) = \{y | x \not\prec E - y\} = \bigcap \{V | x \prec V\}.$$

Then $\gamma_< = \{U_<(x) | x \in E\}$ is a cover of E . — Write now $x \sim y$ if and only if $U_<(x) = U_<(y)$.

This is clearly an equivalence relation, and if we still write $\bar{x} = \{y | x \sim y\}$ and $f_<(\bar{x}) = U_<(x)$, then the function $f_<$ will be a one-to-one mapping of the partition $\pi_< = \{\bar{x} | x \in E\}$ onto the cover $\gamma_<$, satisfying $\bar{x} \subseteq f_<(\bar{x})$ for any $\bar{x} \in \pi_<$. Thus we have established the following

Proposition 1. *If $<$ is a biperfect topogenous order on the set E , then $\mathfrak{A}_< = (\gamma_<, \pi_<, f_<)$ is a kernelled covering of E . $\blacksquare$ ²⁾*

Let now a kernelled covering $\mathfrak{A} = (\gamma, \pi, f)$ of E be given. We write $A < B$ iff $x < B$ for any $x \in A$, and $x < B$ iff $x \in P \subseteq f(P) \subseteq B$ for some $P \in \pi$. The relation $< = <_{\mathfrak{A}}$ so defined is a biperfect topogenous order on E ; the verification is straightforward.³⁾ We thus have

Proposition 2. *If $\mathfrak{A}$ is a kernelled covering of the set E , then $<_{\mathfrak{A}}$ is a biperfect topogenous order on E . $\blacksquare$*

The following proposition will show that the mappings $< \rightarrow \mathfrak{A}_<$ and $\mathfrak{A} \rightarrow <_{\mathfrak{A}}$ are one-to-one correspondences, inverse to each other, between the sets of all biperfect topogenous orders and all kernelled coverings on E .

Proposition 3.

- (I) *Let $< \rightarrow \mathfrak{A}_<$ and $\mathfrak{A} \rightarrow <_{\mathfrak{A}}$. If $\mathfrak{A} = \mathfrak{A}_<$, then $<_{\mathfrak{A}} = <$.*
- (II) *Let $\mathfrak{A} \rightarrow <_{\mathfrak{A}}$ and $< \rightarrow \mathfrak{A}_<$. If $< = <_{\mathfrak{A}}$, then $\mathfrak{A}_< = \mathfrak{A}$.*

PROOF. (I) If $\mathfrak{A} = \mathfrak{A}_<$, then each of the following statements is equivalent to the next one:

$$\begin{aligned} x &<_{\mathfrak{A}} B; \\ x \in \bar{y} \subseteq U_<(y) \subseteq B &\text{ for some } y \in E; \\ x \in \bar{x} \subseteq U_<(x) \subseteq B; \\ U_<(x) &\subseteq B; \\ x &< B. \end{aligned}$$

Thus $<_{\mathfrak{A}} = <$ results proved.

(II) It is clear that if $< = <_{\mathfrak{A}}$ for some kernelled covering $\mathfrak{A}$, then in view of “ $x < B$ iff $x \in P \subseteq f(P) \subseteq B$ ” we have $U_<(x) = f(P)$. Clearly, if x runs through E , $f(P)$ will run through γ . This proves $\gamma_< = \gamma$. Also $\pi_< = \pi$, because $x \sim y$, i.e. $U_<(x) = U_<(y)$ iff x and y belong to the same member of the partition π . By what has

²⁾ If the relation $<$ is a semi-topogenous order (i.e. if it is supposed to satisfy only the first three of the five conditions listed in Definition 1), then the two formulae used to define $U_<(x)$ fail to be equivalent. This yields two possibilities for the definition of $U_<(x)$, the validity of Proposition 1. being preserved by each of the two.

³⁾ Of course, π being a partition, the set $P \in \pi$ is uniquely determined by x , a fact needed in establishing property $(B_{\cap})$.

already been said, we clearly have also $f_{<} = f$, since to a given member of $\pi_{<} = \pi$ there belongs the same $f(P) = U_{<}(x)$ according to $\mathfrak{A}$ and to $\mathfrak{A}_{<} = \mathfrak{A}(<_{\mathfrak{A}})$. ■

Thus we have proved the two notions of biperfect topogenous order and kernelled covering to be equivalent.

Definition 3. A biperfect topogenous order $<$ on a set E is subordinated to a covering γ of E , if $< = <_{\mathfrak{A}}$ for some kernelled covering $\mathfrak{A} = (\alpha, \pi, f)$ such that $\alpha \cong \gamma^4$.

A covering γ is superposed to the biperfect topogenous order $<$, if $<$ is subordinated to γ . ■

One easily sees that the symmetrical perfect topogenous order defined by

$$A <_{\gamma} B \text{ if and only if } St(A, \gamma) \subseteq B$$

is the intersection of all the biperfect topogenous orders subordinated to γ .

As a matter of fact, let $<$ be the intersection of all the biperfect topogenous orders subordinated to γ . Then

$$A < B \text{ iff } A <_{\mathfrak{A}} B \text{ for each } \mathfrak{A} = (\alpha, \pi, f) \text{ with } \alpha \cong \gamma$$

Equivalently, $A < B$ iff for each $x \in A$ and any $G \in \gamma$ satisfying $x \in G$, one has $x \in G \subseteq B$.⁵⁾ Thus

$$A < B \text{ iff } \bigcup_{x \in A} St(x, \gamma) = St(A, \gamma) \subseteq B,$$

i.e. $< = <_{\gamma}$. ■

The one-to-one correspondence existing between biperfect topogenous orders and kernelled coverings over a given set E makes it natural to adopt the following

Definition 4. If $\mathfrak{A}_1 = (\alpha_1, \pi_1, f_1)$ and $\mathfrak{A}_2 = (\alpha_2, \pi_2, f_2)$ are kernelled covers and $<_{\mathfrak{A}_1}$ and $<_{\mathfrak{A}_2}$ are the corresponding biperfect topogenous orders over a set E , then

$$\mathfrak{A}_1 \cong \mathfrak{A}_2 \text{ if and only if } <_{\mathfrak{A}_1} \subseteq <_{\mathfrak{A}_2}. \quad \blacksquare^6)$$

If we wish to obtain an "inner" characterization of the partial order thus introduced, we arrive at the following

Proposition 4. $\mathfrak{A}_1 \cong \mathfrak{A}_2$ iff for any $P \in \pi_1$ and $Q \in \pi_2$, $P \cap Q \neq \emptyset$ implies $f_2(Q) \subseteq f_1(P)$.

Corollary. If $\mathfrak{A}_1 \cong \mathfrak{A}_2$, then $\alpha_2 \cong \alpha_1$ for the first components.

PROOF. The condition is necessary:

Let $\mathfrak{A}_1 \cong \mathfrak{A}_2$. If $P \cap Q \neq \emptyset$ ($P \in \pi_1, Q \in \pi_2$), then $x \in P \cap Q$ for some $x \in E$, and $x \in P \subseteq f_1(P)$ implies $x \in f_1(P)$.

Then, however, $x \in f_2(Q)$, i.e. $x \in Q' \subseteq f_2(Q') \subseteq f_1(P)$ for some $Q' \in \pi_2$. But π_2 is a partition of E , so $Q' = Q$ and $f_2(Q) \subseteq f_1(P)$.

The condition is sufficient:

⁴⁾ I.e. for $A \in \alpha$ there is $G \in \gamma$ such that $A \subseteq G$.

⁵⁾ This in view of the "selection-function" aspect of the notion of kernelled covering.

⁶⁾ Sometimes, for simplicity' sake we write $<_1$ instead of $<_{\mathfrak{A}_1}$.

Suppose it is satisfied, and let $x <_1 B$, i.e. $x \in P \subseteq f_1(P) \subseteq B$ for some $P \in \pi_1$. Now for the uniquely determined $Q \in \pi_2$ satisfying $x \in Q$ we have $x \in Q \subseteq f_2(Q) \subseteq f_1(P) \subseteq B$, hence $x <_2 B$. By the biperfectness of $<_2$ this yields $<_1 \subseteq <_2$, i.e. $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$.

This completes the proof of the proposition.

Now let $\mathfrak{A}_1 \subseteq \mathfrak{A}_2$. Since f_2 is an onto mapping, each member of α_2 can be written in the form $f_2(Q)$, $Q \in \pi_2$, and if $x \in Q$ then for $x \in P \in \pi_1$ we have $f_2(Q) \subseteq f_1(P) \in \alpha_1$. Hence $\alpha_2 \subseteq \alpha_1$. ■

In the same manner as one partially orders the kernelled covers of a given set by referring to the corresponding biperfect topogenous orders, one can also define unions and intersections of kernelled covers with the help of their biperfect topogenous counterparts.

References

- [1] Á. CSÁSZÁR, Foundations of general topology, Oxford, 1963.

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