

Radius of convexity of convex sum of univalent functions

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Introduction

Let S denote the class of functions $f(z)$, normalized so that $f(0)=0$, $f'(0)=1$, that are regular univalent in E , the open unit disc. For a fixed real number λ , $0 < \lambda < 1$, consider the function $h_\lambda(z)$ given by

$$(1) \quad h_\lambda(z) = (1-\lambda)z + \lambda f(z).$$

TRIMBLE [4] showed that if f is convex then $h_\lambda(z)$ is close-to-convex (and hence univalent) in E for all admissible values of λ and if $\lambda \geq \frac{2}{3}$, then $h_\lambda(z)$ is starlike univalent in E .

Let J denote the class of functions $f(z)$ normalized as above, for which $\operatorname{Re} f'(z) > 0$ for z in E . Then it is well known [3] that J is a proper subclass of S . In a recent paper [1], the author considered the class of function $h_\lambda(z)$ defined by (1) when $f \in J$ and obtained its radius of convexity. In this paper we propose to consider a similar problem. Let J_B denote the subclass of J whose functions $f(z)$ are characterized by the property that

$$(2) \quad |f'(z) - 1| < 1 \quad z \in E.$$

Let $h_\lambda(z)$ be defined by (1) and $f \in J_B$. Then we obtain the radius of convexity of the class of functions $h_\lambda(z)$ so defined. It may be remarked that while in [1], ZMO-ROVIC's technique [5] was found to work well, here a different method had to be adopted.

Radius of convexity

Theorem For a fixed λ , $0 < \lambda < 1$ let $h_\lambda(z)$ be defined by (1) and $f \in J_B$. Then $h_\lambda(z)$ is convex in

$$(3) \quad |z| < r_\lambda = \begin{cases} \left(\frac{\sqrt{(1-\lambda)(1+3\lambda)} - (1-\lambda)}{2\lambda} \right)^{1/2} & \text{for } 0 < \lambda \leq \frac{\sqrt{5}+1}{4}, \\ \frac{1}{2\lambda} & \text{for } \frac{\sqrt{5}+1}{4} \leq \lambda < 1. \end{cases}$$

These results are sharp in the sense that for each λ , there exists a function $h_\lambda(z)$ which is not convex in a larger circle.

PROOF. Since the function $\psi(z) = f'(z) - 1$ is bounded by one, $\psi(0) = 0$, we have by Schwarz lemma

$$|f'(z) - 1| = |\psi(z)| \leq |z|.$$

Thus we may write

$$(4) \quad f'(z) - 1 = z\Phi(z)$$

where $\Phi(z)$ is regular and $|\Phi(z)| \leq 1$ in E . Differentiating (1) and substituting the value of $f'(z)$ from (4), we obtain

$$(5) \quad h'_\lambda(z) = 1 + \lambda z\Phi(z)$$

so that

$$(6) \quad \begin{aligned} \operatorname{Re} \left\{ 1 + \frac{zh''_\lambda(z)}{h'_\lambda(z)} \right\} &= \operatorname{Re} \left\{ 1 + \frac{\lambda z(\Phi(z) + z\Phi'(z))}{1 + \lambda z\Phi(z)} \right\} \cong \\ &\cong 1 - \max_{|z|=r} \max_{\Phi} \lambda r \left\{ \frac{|\Phi| + r|\Phi'|}{1 - \lambda r|\Phi|} \right\} \cong 1 - \max_{\kappa} \lambda r \left\{ \frac{(1-r^2)\kappa + r(1-\kappa^2)}{(1-r^2)(1-\lambda r\kappa)} \right\}, \\ &\quad r = |z|, \kappa = |\Phi|, 0 \leq \kappa \leq 1, \end{aligned}$$

where we have made use of the well-known fact that ([2])

$$|\Phi'| \leq \frac{1 - |\Phi|^2}{1 - r^2}.$$

To maximize the second term on the right of (6), let us put

$$\Psi = \frac{(1-r^2)\kappa + r(1-\kappa^2)}{1 - \lambda r\kappa}$$

which yields on differentiation

$$\frac{d\Psi}{d\kappa} = \frac{\lambda r^2 \kappa^2 - 2r\kappa + 1 - r^2 + \lambda r^2}{(1 - \lambda r\kappa)^2}$$

so that the absolute maximum of Ψ occurs for

$$(7) \quad \kappa = \kappa_0 = \frac{1 - \sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r}.$$

It is easy to verify that $\frac{d^2\Psi}{d\kappa^2} < 0$ so that we have indeed a maximum at (7). Also, κ_0 is clearly positive but not always less than or equal to one. So we have to consider the case when $\kappa_0 \notin [0, 1]$. In this case Ψ is monotonic increasing so that the maximum value of Ψ occurs at $\kappa = 1$.

Taking $\kappa = 1$, we obtain from (6)

$$\operatorname{Re} \left\{ 1 + \frac{zh''_\lambda(z)}{h'_\lambda(z)} \right\} \cong 1 - \frac{\lambda r}{1 - \lambda r} = \frac{1 - 2\lambda r}{1 - \lambda r}$$

so that $h_\lambda(z)$ is convex in $|z| < r_\lambda$ where r_λ is the smallest positive root of the equation

$$1 - 2\lambda r = 0.$$

This gives

$$(8) \quad r\lambda = \frac{1}{2\lambda}.$$

Since $r_\lambda \leq 1$, we must have $\lambda \geq \frac{1}{2}$ so that formula (8) is not valid for $\lambda < \frac{1}{2}$.

Taking $z = z_0 = \frac{1 - \sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r}$, we obtain from (6)

$$\operatorname{Re} \left\{ 1 + \frac{zh''_\lambda(z)}{h'_\lambda(z)} \right\} \geq \frac{2 \left(\frac{1 - \sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda} \right) (1 + \lambda r^2 - \lambda) - 2\lambda r^2}{(1-r^2)(1-\lambda r z)}.$$

Therefore $h_\lambda(z)$ is convex in $|z| < r_\lambda$ where r_λ is the smallest positive root of the equation

$$(9) \quad (1 - \sqrt{(1-\lambda)(1+\lambda r^2)})(1 + \lambda r^2 - \lambda) - \lambda^2 r^2 = 0$$

that is,

$$\lambda r^4 + r^2(1-\lambda) - (1-\lambda) = 0.$$

This gives

$$(10) \quad r_\lambda = \left(\frac{\sqrt{(1-\lambda)(1+3\lambda)} - (1-\lambda)}{2\lambda} \right)^{1/2} = b \quad (\text{say}).$$

The transition from formula (10) to formula (8) occurs for that value of λ for which z_0 given by (7) = 1. This gives

$$(11) \quad \frac{1 - \sqrt{(1-\lambda)(1+\lambda r^2)}}{\lambda r} = 1$$

Solving (11) with the help of (8), that is, taking $r = \frac{1}{2\lambda}$, we obtain $\lambda = \frac{\sqrt{5}+1}{4}$.

So we have to use (10) for $0 < \lambda \leq \frac{\sqrt{5}+1}{4}$ and (8) for $\frac{\sqrt{5}+1}{4} \leq \lambda < 1$.

We now determine the form of the extremal functions. In case (8) the extremal function is easily seen to be

$$(12) \quad h_\lambda(z) = (1-\lambda)z + \lambda \left(z + \frac{1}{2} z^2 \right) = z + \frac{\lambda}{2} z^2 - \frac{z-\beta}{1-z\beta}$$

In case (10) we obtain the extremal function thus: We take $\Phi(z) = -\frac{z-\beta}{1-z\beta}$ in (5) and

$$(13) \quad \beta = \frac{1-\lambda-\lambda b^2}{(1-\lambda)b},$$

where b is given by (10). Then $h_\lambda(z)$ is given by

$$(14) \quad h'_\lambda(z) = 1 - \frac{\lambda z(z-\beta)}{1-z\beta}.$$

From (14) we obtain

$$(15) \quad 1 + \frac{zh''_{\lambda}(z)}{h'_{\lambda}(z)} = \frac{\beta^2 z^2 (1 - \lambda) + 2\beta z (\lambda - 1 + \lambda z^2) + 1 - \lambda z^2 - 2\lambda z^2}{(1 - \beta z)(1 - \beta z + \lambda \beta z - \lambda z^2)}.$$

The numerator of the right side of (15) vanishes for $z=b$ and β given by (13) so that $h_{\lambda}(z)$ is not convex in a circle larger than $|z| < b$.

We have yet to prove that $|\beta| \leq 1$ in order that Φ be bounded in E . That is, we must show that

$$(16) \quad -1 \leq \frac{1 - \lambda - \lambda b^2}{(1 - \lambda)b} \leq 1.$$

From (10) we deduce that $\lambda = \frac{1 - b^2}{1 - b^2 + b^4}$ so that the right inequality in (16) will hold if

$$\frac{1 - b^2}{1 - b^2 + b^4} \geq \frac{1 - b}{1 - b + b^2}$$

that is, if $1 + b \geq b^2$, so the right inequality holds. The left inequality will hold if we show that

$$\frac{1 - b^2}{1 - b^2 + b^4} \leq \frac{1 + b}{1 + b + b^2}$$

that is, if

$$(17) \quad b \geq \frac{\sqrt{5} - 1}{2}.$$

Since for $0 < \lambda \leq \frac{\sqrt{5} + 1}{4}$, $b \geq \frac{\sqrt{5} - 1}{2}$, (17) always holds. This completes the proof of the theorem.

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