

## Gauss and Codazzi equations in a subspace of a Kähler space

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### 1. Preliminaries

In this paper we shall obtain Gauss and Codazzi equations in a subspace of a Kähler space. Let us consider a Kähler space  $K_{2n}$  covered by a system of complex co-ordinate neighbourhoods.

$$(1.1) \quad z^\alpha = \zeta^\alpha + i\zeta^{\bar{\alpha}} \quad *)$$

and a subspace  $K_{2m}$  ( $m < n$ ) defined by

$$(1.2) \quad z^\alpha = \zeta^\alpha(\eta^a) + i\zeta^{\bar{\alpha}}(\eta^a) \quad **)$$

where  $\eta^a$  are parameters on the subspace.

Introduce a complex co-ordinate system ( $u^\alpha$ ) defined by

$$(1.3) \quad u^\alpha = \eta^\alpha + i\eta^{\bar{\alpha}}$$

Representing both the Kähler space  $K_{2n}$  and the analytic subspace  $K_{2m}$  by the real co-ordinates  $\zeta^h$  and  $\eta^a$  respectively, we have

$$(1.4) \quad \zeta^h = \zeta^h(\eta^a)$$

The tensor  $'F_b^a$  is such that

$$(1.5) \quad F_i^h B_b^i = 'F_b^a B_a^h$$

where  $B_b^h = \partial_b \zeta^h$

and

$$(1.6) \quad 'F_c^b 'F_b^a = -A_c^a$$

The induced fundamental metric  $'g_{ab}$  is given by

$$(1.7) \quad 'g_{cb} = B_c^j B_b^i 'g_{ji}$$

and

$$(1.8) \quad 'g_{cb} = 'F_c^f 'F_b^e 'g_{fe}$$

\*) The notations used in this paper are from Yano's book

\*\*\*) The indices  $a, b, c, \dots$  take values from  $1, 2, \dots, m$ ,  $\bar{1}, \bar{2}, \dots, \bar{m}$ , and  $\alpha, \beta, \dots = 1, 2, \dots, m$ .

The Waerden—Bortolotti derivative of  $B_b^h$  is given by ([1], p. 103)

$$(1.9) \quad \nabla_c B_b^h = \partial_c B_b^h + B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\}$$

which is a vector of  $K_{2n}$ , orthogonal to  $K_{2m}$ .

Let us denote  $2n-2m$  mutually orthogonal unit normals to  $K_{2m}$  by  $C_x^h$ , where

$$x, y, z = m+1, \dots, n, \overline{m+1}, \dots, \overline{n}.$$

Thus ([1], p. 108)

$$(1.10) \quad \nabla_c B_b^h = H_{cbx} C_x^h$$

where  $H_{cbx}$  are the second fundamental tensors of  $K_{2m}$  with respect to the normals  $C_x^h$ .  $C_x^h$  satisfies the following relations

$$(1.11) a \quad B_c^j C_x^i g_{ji} = 0,$$

$$(1.11) b \quad C_y^j C_x^i g_{ji} \delta_{yx}.$$

The equation of Weingarten are given by ([1], p. 106)

$$(1.12) \quad \nabla_c C_x^h = -H_{cx}^a B_a^h + L_{cxy} C_y^h,$$

where

$$L_{cxy} = (\nabla_c C_x^j) C_y^i g_{ji}.$$

## 2. Gauss and Codazzi Equations

Taking Waerden—Bortolotti derivative of (1.9) we get

$$(2.1) \quad \begin{aligned} \nabla_d \nabla_c B_b^h &= \nabla_d (\partial_c B_b^h) + \nabla_d \left( B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \right) - \nabla_d B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} = \\ &= \partial_d \partial_c B_b^h + \partial_c B_b^i B_d^f \left\{ \begin{matrix} h \\ f \ l \end{matrix} \right\} - \partial_l B_b^h \left\{ \begin{matrix} l \\ d \ c \end{matrix} \right\} - \partial_c B_l^h \left\{ \begin{matrix} l \\ d \ b \end{matrix} \right\} + \\ &\quad + \left[ B_c^j (\partial_d B_b^i) \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} + B_b^i (\partial_d B_c^j) \left\{ \begin{matrix} h \\ i \ j \end{matrix} \right\} + B_c^j B_b^i \partial_d \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} + \right. \\ &\quad \left. + B_c^j B_b^i \left\{ \begin{matrix} l \\ j \ i \end{matrix} \right\} \left\{ \begin{matrix} h \\ l \ d \end{matrix} \right\} - B_l^j B_b^i \left\{ \begin{matrix} h \\ i \ j \end{matrix} \right\} \left\{ \begin{matrix} l \\ c \ d \end{matrix} \right\} - B_c^j B_l^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ d \end{matrix} \right\} \right] - \\ &\quad - \left[ \partial_d B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} + \partial_d \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} B_a^h + B_a^i B_d^f \left\{ \begin{matrix} h \\ f \ l \end{matrix} \right\} \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} - \right. \\ &\quad \left. - B_a^h \left\{ \begin{matrix} a \\ l \ b \end{matrix} \right\} \left\{ \begin{matrix} l \\ d \ c \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ l \end{matrix} \right\} \left\{ \begin{matrix} l \\ b \ d \end{matrix} \right\} \right]. \end{aligned}$$

In (2.1), on changing the indices  $c, d$  and on subtracting it from the new equation, we obtain the following on simplification

$$(2.2) \quad \nabla_d \nabla_c B_b^h - \nabla_c \nabla_d B_b^h = B_a^h R_{dcb}^a + B_b^i B_c^j K_{jdi}^h - B_b^i S_{cjd}^h,$$

where we have put

$$\begin{aligned} {}'R^a{}_{dcb} &= \partial_c \begin{Bmatrix} a \\ d \ b \end{Bmatrix} - \partial_d \begin{Bmatrix} a \\ c \ b \end{Bmatrix} + \begin{Bmatrix} a \\ c \ l \end{Bmatrix} \begin{Bmatrix} l \\ b \ d \end{Bmatrix} - \begin{Bmatrix} a \\ d \ l \end{Bmatrix} \begin{Bmatrix} l \\ b \ c \end{Bmatrix}, \\ K_{jdi}{}^h &= \partial_d \begin{Bmatrix} h \\ j \ i \end{Bmatrix} - \partial_j \begin{Bmatrix} h \\ d \ i \end{Bmatrix} + \begin{Bmatrix} h \\ l \ d \end{Bmatrix} \begin{Bmatrix} l \\ j \ i \end{Bmatrix} - \begin{Bmatrix} h \\ j \ l \end{Bmatrix} \begin{Bmatrix} l \\ d \ i \end{Bmatrix}, \end{aligned}$$

and

$$S_{cjd}{}^h \stackrel{\text{def}}{=} B_c{}^j \partial_j \begin{Bmatrix} h \\ d \ i \end{Bmatrix} + B_c{}^j \begin{Bmatrix} h \\ j \ l \end{Bmatrix} \begin{Bmatrix} l \\ d \ i \end{Bmatrix} + B_d{}^j \partial_c \begin{Bmatrix} h \\ j \ i \end{Bmatrix} + B_d{}^j \begin{Bmatrix} l \\ j \ i \end{Bmatrix} \begin{Bmatrix} h \\ l \ c \end{Bmatrix}.$$

Waerden—Bortolotti derivative of (1. 10) yields

$$(2.3) \quad {}'\nabla_d {}'\nabla_c B_b{}^h = H_{cbx} {}'\nabla_d C_x{}^h + C_x{}^h {}'\nabla_d H_{cbx}.$$

By changing  $c, d$  and subtracting it from the resulting equation so obtained, we get

$$(2.4) \quad {}'\nabla_d {}'\nabla_c B_b{}^h - {}'\nabla_c {}'\nabla_d B_b{}^h = C_x{}^h ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) + H_{cbx} {}'\nabla_d C_x{}^h - H_{dbx} {}'\nabla_c C_x{}^h.$$

Using (1. 12), (2. 4) reduces to

$$(2.5) \quad \begin{aligned} &{}'\nabla_d {}'\nabla_c B_b{}^h - {}'\nabla_c {}'\nabla_d B_b{}^h = \\ &= C_x{}^h ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) - B_a{}^h (H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + C_y{}^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}). \end{aligned}$$

From (2. 2) and (2. 5) we get

$$(2.6) \quad \begin{aligned} &B_a{}^h {}'R^a{}_{dcb} + B_b{}^i B_c{}^j K_{jdi}{}^h - B_b{}^i S_{cjd}{}^h = \\ &= C_x{}^h ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) - B_a{}^h (H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + C_y{}^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}). \end{aligned}$$

Multiplying both sides of (2. 6) by  $g_{kh} B_e{}^k$  and using (1. 7) and (1. 11)a, we obtain

$$(2.7) \quad {}'g_{ea} ({}'R^a{}_{dcb} + H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + B_b{}^i B_c{}^j B_e{}^k g_{kh} K_{jdi}{}^h - B_b{}^i B_e{}^k g_{kh} S_{cjd}{}^h = 0$$

or

$$(2.8) \quad {}'R_{dcb}{}^e = H_{dbx} H_{cxe} - H_{cbx} H_{dxe} - B_b{}^i B_c{}^j K_{jdik} + B_b{}^i B_e{}^k S_{cjdk}.$$

(2. 8) will be called the “generalised Gauss characteristic equation” in a subspace of a Kähler space.

Again multiplying (2. 6) by  $g_{kh} C_z{}^k$  and using (1. 11) a, b we get

$$(2.9) \quad \delta_{xz} ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) + \delta_{yz} (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}) = 0$$

We call this the “generalised Mainardi Codazzi relation” in a subspace of a Kähler space.

### 3. Some properties of the second fundamental tensor $H_{cbx}$ .

Let a holomorphic plane element determined by a certain vector  $u^h$  and its transform  $F_i{}^h u^i$  by  $F_i{}^h$  at a point be given. We draw a totally geodesic subspace which is tangent to this holomorphic plane element and passes through this point.

We represent the subspace by

$$(3.1) \quad \xi^h = \zeta^h(\eta^a)$$

where  $a, b, c, d=1, 2$ .

Then the subspace being totally geodesic, we have (1, p. 107)

$$(3.2) \quad \partial_c B_b^h + B_c^j B_b^i \left\{ \begin{matrix} h \\ j \ i \end{matrix} \right\} - B_a^h \left\{ \begin{matrix} a \\ c \ b \end{matrix} \right\} = 0$$

substituting (3.2) in (1.9), we obtain

$$(3.3) \quad {}'\nabla_c B_b^h = 0$$

from which we have

**Theorem 3.1.** *A necessary and sufficient condition for  $B_b^h$  in a subspace of a Kähler space to be totally geodesic is that its Waerden—Bortolotti derivative be zero*

Applying (3.3) in (2.5) we get

$$(3.4) \quad 0 = C_x^h ({}'\nabla_d H_{cbx} - {}'\nabla_c H_{dbx}) - B_a^h (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) + C_y^h (H_{cbx} L_{dxy} - H_{dbx} L_{cxy}).$$

Multiplying both sides of (3.4) by  $g_{kh} B_e^k$  and using (1.7) and (1.11)a we obtain

$${}'g_{ea} (H_{cbx} H_{dx}^a - H_{dbx} H_{cx}^a) = 0$$

or

$$(3.5) \quad H_{cbx} H_{dxe} - H_{dbx} H_{cxe} = 0$$

Thus we have

**Theorem 3.2.** (3.5) is the necessary condition that a subspace of a Kähler space be totally geodesic.

From (1.11) and (3.3) we get

$$(3.6) \quad H_{cbx} = 0$$

since we assume that  $C_x^h \neq 0$ .

Thus we have

**Theorem 3.3.** *A necessary and sufficient condition for a subspace of a Kähler space to be totally geodesic is given by*

$$H_{cbx} = 0$$

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### Reference

- [1] K. YANO, Differential geometry on complex and almost complex spaces, New York, 1965.

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