

Note on differential equations with constant coefficients

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In a previous paper [1], we studied the resonance case in the differential equation of the n^{th} order

$$(a) \quad L[x] \equiv x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \dots + a_{n-j}(t)x^{(j)} = f(t) \quad (1 \leq j \leq n-1),$$

where the coefficients $a_\mu(t)$ ($\mu = 1, 2, \dots, n-j$) and $f(t)$ are continuous and periodic functions of the same period p and

$$(1) \quad a_{n-j}(t) \not\equiv 0.$$

In this note we shall assume that *all the coefficients $a_\mu(t)$ in (a) are constants and $f(t)$ has the period p* . The main results in [1] will be reduced in this case to interesting results.

Setting $\hat{x}(t) = x^{(j)}(t)$ in (a), we obtain the reduced differential equation of order $(n-j)$

$$(\hat{a}) \quad \hat{L}[\hat{x}] \equiv \hat{x}^{(n-j)} + a_1 \hat{x}^{(n-j-1)} + \dots + a_{n-j} \hat{x} = f(t).$$

The homogeneous and adjoint equations corresponding to $(\hat{a})$ are

$$(\hat{b}) \quad \hat{L}[\hat{y}] \equiv \hat{y}^{(n-j)} + a_1 \hat{y}^{(n-j-1)} + \dots + a_{n-j} \hat{y} = 0$$

and

$$(\hat{c}) \quad \bar{L}[\hat{z}] \equiv (-1)^{n-j} \hat{z}^{(n-j)} + (-1)^{n-j-1} a_1 \hat{z}^{(n-j-1)} + \dots + a_{n-j} \hat{z} = 0$$

respectively.

We state the following lemma, which can be proved easily.

Lemma 1. *Under the assumption (1), all the p -periodic solutions of $(\hat{b})$ and $(\hat{c})$ have the mean value zero.*

We solve now the equation $(\hat{b})$ by means of the substitution $\hat{y}(t) = e^{\alpha t}$. Let α_v^* ($v=1, \dots, s$) be the pairwise distinct roots of the characteristic equation corresponding to $(\hat{b})$ with the multiplicities m_v . Equation $(\hat{b})$ has p -periodic solutions iff between the characteristic roots α_v^* (for $v=1, \dots, s$) there exists integral multivalued of $\frac{2\pi i}{p}$. Let the characteristic roots be arranged such that $\alpha_1^*, \dots, \alpha_q^*$ be integral multivalued of $\frac{2\pi i}{p}$, while the others are not. We notice that $\alpha_v^* \neq 0$, other-

wise the equation $(\hat{b})$ possesses a constant solution, which contradicts with the assumption (1) (Lemma 1). We set $\alpha_v^* = \alpha_v + i\beta_v$ ($v=1, \dots, \varrho, \dots, s$). The general solution of $(\hat{b})$ is

$$\hat{y}(t) = \sum_{v=1}^s \sum_{\mu=0}^{\hat{m}_v-1} e^{\alpha_v^* t} {}^v c_\mu t^\mu,$$

where ${}^v c_\mu$ are arbitrary constants. Accordingly the first row of the fundamental matrix solution $\hat{Y}(t)$ of $(\hat{b})$, which represents the n linear independent solutions of $(\hat{b})$, has the form

$$(2) \quad \hat{y}_1^T(t) = (\hat{y}_{1,1}^T(t), \hat{y}_{1,2}^T(t), \dots, \hat{y}_{1,s}(t)).$$

Here ($v=1, \dots, s$)

$$(3) \quad \begin{aligned} \hat{y}_{1,v}^T(t) &= e^{i\beta_v t} \left(1, t, \dots, \frac{t^{\hat{m}_v-1}}{(\hat{m}_v-1)!} \right) e^{\alpha_v t} = \\ &= (e^{i\beta_v t}, O_2, \dots, O_{\hat{m}_v}) \cdot \begin{pmatrix} 1, t, \frac{t^2}{2!}, \dots, \frac{t^{\hat{m}_v-1}}{(\hat{m}_v-1)!} \\ 1, t, \dots, \frac{t^{\hat{m}_v-2}}{(\hat{m}_v-2)!} \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \cdot e^{\alpha_v t} = \hat{\phi}_{1,v}^T(t) e^{\hat{K}_v t}, \end{aligned}$$

where—as it can be easily verified (see e.g. [2] or [3])—

$$(4) \quad \hat{K}_v = \begin{pmatrix} \alpha_v & 1 & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & 1 \\ & & & & \alpha_v \end{pmatrix} \quad \text{and} \quad \hat{\phi}_{1,v}^T(t) = (e^{i\beta_v t}, O_2, \dots, O_{\hat{m}_v}).$$

Consequently the fundamental matrix solution $\hat{Y}(t)$ of $(\hat{b})$ is obtained in the required form

$$(5) \quad \hat{Y}(t) = \hat{\Phi}(t) e^{\hat{K}t}$$

(see [1], § 2), where the matrix $\hat{\Phi}(t)$ is p -periodic and the constant matrix $\hat{K}$ is in the Jordan canonical normal form with the submatrices $\hat{K}_v$ ($v=1, \dots, s$) of order $\hat{m}_v$. Denoting the elements of the first row of $\hat{\Phi}(t)$ by $\hat{\phi}_\mu(t)$ (for $\mu = (v), (v)+1, \dots, (v)+\hat{m}_v-1$; $v=1, \dots, \varrho, \dots, s$), where the index (v) is defined by

$$(6) \quad (v) = \sum_{\mu=1}^{v-1} m_\mu + 1,$$

we see from (4) that

$$(7) \quad \hat{\phi}_{(v)}(t) = e^{i\beta_v t}, \quad \hat{\phi}_\mu(t) \equiv 0 \quad \text{for } \mu = (v)+1, \dots, (v)+\hat{m}_v-1; \quad v=1, \dots, \varrho.$$

The fact that $\hat{y}_{(v)}(t) = \hat{\phi}_{(v)}(t)$ ($v=1, \dots, \varrho$) possesses the mean value zero is also already proved in Lemma 1.

Thus we have proved the following

Theorem 1. *In the case of differential equations with constant coefficients, all the p -periodic functions $\hat{\phi}_\mu(t)$ for $\mu = (v), \dots, (v) + \hat{m}_v - 1$; $v=1, \dots, \varrho$ (see (5) and (7)) have the mean value zero, where $\hat{\phi}_\mu(t)$ denotes the elements of the first row of the matrix $\hat{\Phi}(t)$.*

We turn now to the question of the power order of the solution $x(t)$ of (a) and its first $(n-1)$ derivatives.

It is well known [1], that in the resonance case¹⁾, under the essential condition (1), the solution $\hat{x}(t) = x^{(j)}(t)$ of ($\hat{a}$) and its derivatives $x^{(j+1)}, \dots, x^{(n-1)}$ take on — independent of the initial values — values of the same minimal order

$$(8) \quad \hat{m} = \underset{\substack{(v \text{ Resonance}) \\ v=1, \dots, \varrho}}{\text{Max}} (\hat{m}_v).$$

Further it is also shown in [1], that the minimal order of the solution $x(t)$ of (a) is

$$(9) \quad m = \underset{\substack{(v \text{ Resonance}) \\ v=0, 1, \dots, \varrho}}{\text{Max}} (m_v),$$

where²⁾ the resonance indices $v=1, \dots, \varrho$ are in both equations (a) and ($\hat{a}$) the same and $m_v = m_v(j)$ is obtained from the formula

$$(10) \quad \begin{cases} m_v = \hat{m}_v + \text{Min}(j, i_{v+1}) - \text{Min}(j, i_v) & \text{for } (v=1, \dots, \lambda-1) \\ m_\lambda = \hat{m}_\lambda + j - \text{Min}(j, i_\lambda) \\ m_v = \hat{m}_v & \text{for } (v=\lambda+1, \dots, s), \quad m_0 = \text{Min}(j, i_1). \end{cases}$$

The same statement holds also for the derivatives $x^{(k)}(t)$ ($k=1, \dots, j-1$), if the index j is replaced by $j-k$ in the formula (10).

Referring to theorem 1 and the definition of the index i_v and λ (see [1], § 3), we obtain the following corollaries:

1. In the case of differential equations with constant coefficients, there do not exist such indices i_v .

2. $m_v = \hat{m}_v$ (for $v=1, \dots, \varrho, \dots, s$), $m_0 = j$. This follows from (10) and corollary 1.

Referring to (9), it is required for $j > \hat{m}$ to know whether the index $v=0$ is a resonance index or not.

We state the following

Lemma 2. $v=0$ is a resonance index of the differential equation (a) iff the p -periodic function $f(t)$ has a mean value different from zero.

We evaluate the additional p -periodic solution $z_0(t)$ of the adjoint equation corresponding to (a) ([1], § 6). This p -periodic solution satisfies the inhomogeneous adjoint reduced differential equation

$$(11) \quad \tilde{L}[\hat{u}] \equiv (-1)^{n-j} \hat{u}^{(n-j)} + (-1)^{n-j-1} a_1 \hat{u}^{(n-j-1)} + \dots + a_{n-j} \hat{u} = 1$$

¹⁾ For the definition of the resonance case or resonance index see [1], § 7.

²⁾ For more investigations on the index $v=0$, see [1], § 2 theorem 1 and § 6 theorem 8.

(see [4]), which possesses a particular solution $z_0^*(t) = \frac{1}{a_{n-j}}$. Thus $z_0(t)$ is uniquely determined up to an additive linear combination of the p -periodic solutions $\hat{z}(t)$ of (c), which have — by virtue of Lemma 1 — the mean value zero. However it can be shown that z_0 is a constant.

Referring to the formula (9), corollary 2 and lemma 2, we obtain the following

Theorem 2. If $\int_0^p f(t)dt=0$, then the solution $x(t)$ of (a) and all its first $(n-1)$ derivatives take on in the resonance case — independent of the initial values — the same minimal power order, i.e. t^m with $m = \hat{m} = \underset{\substack{\text{v Resonance} \\ v=1, \dots, q}}{\text{Max}} (\hat{m}_v)$. Further if $\int_0^p f(t)dt \neq 0$ and simultaneously $j > m$, then the minimal power order of the derivatives $x^{(k)}(t)$ ($k = 0, 1, \dots, j-m$) decreases monotonically by 1 starting from $x(t)$ with the minimal power order t^j till $x^{(j-m)}(t)$ with the minimal power order t^m , and remain from this value constant and equal t^m . Finally if $\int_0^p f(t)dt \neq 0$ and $j \leq m$, then the first statement holds.

References

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