

Decompositions of the Fredholm Operators

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We shall use the following notations:

H is a separable Hilbert space with infinite dimension.

$L(H)$ is the Banach algebra of all continuous linear transformations of H ,

$Gl(H)$ is the group of the invertible elements in $L(H)$.

$\text{Ker } A = \{\varphi \mid \varphi \in H, A\varphi = 0\}$ is the kernel of $A \in L(H)$ $\text{Coker } A = H/\text{Im } A$ where $\text{Im } A$ is the image of the operator A .

If $\text{Im } A$ is a closed subspace in H then $\text{Coker } A$ can be identified with $(\text{Im } A)^\perp$, the orthogonal complement of $\text{Im } A$.

An operator $A \in L(H)$ is called Fredholm operator if both $\dim \text{Ker } A$ and $\dim \text{Coker } A$ are finite. The difference

$$\dim \text{Ker } A - \dim \text{Coker } A$$

is called the index of the operator A .

The index of A and the space of all Fredholm operators will be denoted by $\text{Index } A$, and $\mathcal{F}$, respectively.

We mention that

$$\text{Index } A = \dim H/V - \dim H/A(V)$$

where $V \subset H$ is a subspace satisfying the next conditions:

$$\text{Ker } A \cap V = 0$$

$$\dim H/V < \infty \quad (\text{see [1]}).$$

Let be

$$\mathcal{F}_k = \{A : A \in \mathcal{F}, \text{Index } A = k\}.$$

It was proved that the space $\mathcal{F} \subset L(H)$ is an open set and its components are just the $\mathcal{F}_k$ -s. Furthermore $\mathcal{F}_k \cdot \mathcal{F}_l \subset \mathcal{F}_{k+l}$ which can be expressed as

$$\text{Index } AB = \text{Index } A + \text{Index } B$$

(see [1], [2], [3]).

We deal with the question whether $\mathcal{F}_k \cdot \mathcal{F}_l$ and $\mathcal{F}_{k+l}$ can be identical, and we shall prove the following theorem

Theorem. If $k \geq 0$ or $l \leq 0$ then $\mathcal{F}_k \cdot \mathcal{F}_l = \mathcal{F}_{k+l}$. If $k < 0$ and $l > 0$ then the equality $\mathcal{F}_k \cdot \mathcal{F}_l = \mathcal{F}_{k+l}$ does not hold. An operator $A \in \mathcal{F}_{k+l}$ can be decomposed in the form $A = A_1 A_2$ where $A_1 \in \mathcal{F}_k$, $A_2 \in \mathcal{F}_l$ if and only if

$$\dim \text{Ker } A \geq l$$

or equivalently

$$\dim \text{Coker } A \geq -k.$$

PROOF. Let us take an orthonormal basis $\{e_i\}_{i=1}^\infty$ of the Hilbert space H , and define the following operators

$$T_k e_i = \begin{cases} e_{i-k} & \text{if } i > k \\ 0 & \text{if } i \leq k \end{cases}$$

Obviously $\text{Index } T_k = k$ and if $k \geq 0$, then $T_k T_{-k} = \text{id}_H$. If $k \geq 0$ or $l \leq 0$ then an operator $A \in \mathcal{F}_{k+l}$ can be written as a product:

$$A = A_1 A_2 \quad \text{where } A_1 \in \mathcal{F}_k, \quad A_2 \in \mathcal{F}_l.$$

The decomposition is the following:

$$A = T_k (T_{-k} A) \quad \text{or} \quad A = (A T_{-l}) T_l$$

respectively.

If $k < 0$ and $l > 0$ a similar decomposition is not always possible. If $\dim \text{Ker } A < l$, then a decomposition is surely impossible. Namely suppose $A = A_1 A_2$ where $A_1 \in \mathcal{F}_k$, $A_2 \in \mathcal{F}_l$. Then $\dim \text{Ker } A_2 \geq l$ and $l > \dim \text{Ker } A = \dim \text{Ker } A_1 A_2 \geq \dim \text{Ker } A_2 \geq l$ which is a contradiction.

We obtain a necessary condition on the possibility of a decomposition. Clearly the operator T_{k+l} does not satisfy this condition, so in this case the equality $\mathcal{F}_k \mathcal{F}_l = \mathcal{F}_{k+l}$ does not hold.

The condition $\dim \text{Ker } A \geq l$ is equivalent to $\dim \text{Coker } A \geq -k$.

Indeed:

$$\dim \text{Ker } A - l = \dim \text{Coker } A + k \geq 0$$

One of these conditions is, sufficient. Suppose then $\dim \text{Ker } A \geq l$. If $k+l \geq 0$, we choose a subspace $V \subset H$ in the following manner:

V is the orthogonal complement of a subspace of $\text{Ker } A$ having dimension $\dim \text{Ker } A - (k+l)$. If $k+l < 0$ then V is defined to be the orthogonal complement of a subspace of dimension $\dim \text{Ker } A - (k+l)$, which contains $\text{Ker } A$. Since

$$\dim H/\text{Im } A = \dim H/V$$

so there exists an operator $B \in \text{Gl}(H)$ which maps $\text{Im } A$ onto V . For this B we have

$$\text{Ker } BA = \text{Ker } A, \quad \text{Im } BA = V.$$

We decompose H as the direct sum of three orthogonal subspaces

$$\begin{aligned} H &= (\text{Ker } A \cap V^\perp) \oplus \{((\text{Ker } A)^\perp \cap V^\perp) \oplus (\text{Ker } A \cap V)\} \oplus ((\text{Ker } A)^\perp \cap V) = \\ &= H_1 \oplus H_2 \oplus H_3 \end{aligned}$$

Choose an orthonormal basis $\{e_i\}_{i=1}^\infty$ in H such that $\{e_i\}_{i=1}^n$ is a basis for H_1 , $\{e_i\}_{i=n+1}^m$ is a basis for H_2 and finally $\{e_i\}_{i=m+1}^\infty$ is a basis for H_3 .

We define the action of a certain operator on this basis:

$$Ce_i = \begin{cases} (BA|_{(\text{Ker } T)^\perp})^{-1} e_{i-k-l} & \text{if } e_i \notin \text{Ker } A \\ e_i & \text{if } e_i \in \text{Ker } A \end{cases}$$

The definition is correct, since the map

$$BA|_{(\text{Ker } A)^\perp} : (\text{Ker } A)^\perp \rightarrow V$$

is one to one.

Hence a basis of V is mapped onto a basis of $(\text{Ker } A)^\perp$ by the inverse. Furthermore C is identity on $\text{Ker } A$, so $C \in Gl(H)$. Let us compute the action of BAC on the given basis:

First if $e_i \in \text{Ker } A$

$$BACe_i = BA(Ce_i) = BAe_i = 0.$$

Secondly if $e_i \notin \text{Ker } A$ then

$$BACe_i = BA(Ce_i) = BA(BA|_{(\text{Ker } A)^\perp})^{-1} e_{i-k-l} = e_{i-k-l} = T_k T_l e_i = T_{k+l-m} T_m e_i$$

if $i > m \cong l$, where $m = \dim \text{Ker } A$. Thus we obtain:

$$BAC = T_{k+l-m} T_m$$

On the other hand $T_{m-l} T_{l-m} = \text{id}_H$ consequently

$$BAC = T_{k+l-m} T_{m-l} T_{l-m} T_m$$

Here $B, C \in Gl(H)$, so we rewrite this equality:

$$A = (B^{-1} T_{k+l-m} T_{m-l}) (T_{l-m} T_m C^{-1})$$

which is the desired decomposition.

References

- [1] M. F. ATIYAH, Lectures on K-theory, *Cambridge (Mass.)*, 1965.

(Received February 20, 1970.)