

## Weighted Nikolskii-type inequalities II

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**Abstract.** In the paper is proved that the Nikolskii-type inequality (2) below is sharp in the cases:  $p \leq q$ ;  $p > q, m > 1$ ;  $p > q, 0 < m < 1$ .

The present paper is a contribution to the investigations initiated by P. NÉVAI, V. TOTIK and others (see [8], [9] and [1] for further references).

Let

$$w(x) = w_\alpha(x) = |x|^{\alpha/2} \cdot \exp(-|x|^m), \quad x \in \mathbb{R}, \quad m > 0.$$

Given  $p, q$  and  $m$  such that  $0 < p, q \leq \infty, m > 0$  define the Nikolskii constant  $N_n = N_n(m, p, q)$ ,  $n = 1, 2, \dots$  by

$$(1) \quad N_n(m, p, q) = \begin{cases} n^{1/m(1/p-1/q)} & \text{if } p \leq q, \\ n^{(1-1/m)(1/q-1/p)} & \text{if } p > q \text{ and } m > 1, \\ (\log(n+1))^{1/q-1/p} & \text{if } p > q \text{ and } m = 1, \\ 1 & \text{if } p > q \text{ and } 0 < m < 1. \end{cases}$$

For  $0 < p \leq \infty$  denote  $\|f\|_p$  the expression

$$\|f\|_p = \left( \int |f(t)|^p dt \right)^{1/p}.$$

One of the results proved in [1] is the following.

**Theorem** ([1]). Suppose  $0 < p, q \leq \infty, \alpha \geq 0, m > 0$ . Then for any polynomial  $p_n \in \Pi_n$  of degree  $\leq n$  we have

$$(2) \quad \|p_n w_\alpha\|_p \leq c \cdot N_n(m, p, q) \cdot \|p_n w_\alpha\|_q,$$

where  $c = c(m, p, q)$  is a positive constant independent of  $n, p_n$ .

The aim of the present note is to prove that this theorem is sharp in the cases  $p \leq q$ ;  $p > q$  and  $m > 1$ ;  $p > q$  and  $0 < m < 1$ . We think that this theorem is sharp also in the case of  $p > q$  and  $m = 1$ , but now we are not able to prove it. Namely, we show that in the mentioned cases there exist  $c^* > 0$  and polynomials  $\{R_n^*\}_{n=1}^\infty$  with  $\deg R_n^* \leq n$  such that

$$(3) \quad \|R_n^* w_\alpha\|_p \geq c^* N_n(m, p, q) \cdot \|R_n^* w_\alpha\|_q.$$

for  $n = 1, 2, \dots$ .

For the proof of (3) we need some lemmas. First we prove estimates for the Christoffel function of  $w_\alpha(x)$ . For the definition and other results see [2], Ch. 1.

**Lemma 1** ([3], p. 338, Lemma (2.2)). *Let*

$$v^{(\alpha)}(x) = \begin{cases} |x|^\alpha, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \quad \alpha > -1,$$

and denote  $\lambda_n(v^{(\alpha)}, \xi)$  the  $n$ -th Christoffel function of  $v^{(\alpha)}(x)$ . Then

$$(4) \quad \lambda(v^{(\alpha)}, \xi) \asymp n \begin{cases} n^{-2}, & 1 - \frac{c_4}{n^2} \leq \xi^2 \leq 1, \\ \frac{1}{n} |\xi|^\alpha \cdot (1 - \xi^2)^{\frac{1}{2}}, & \frac{c_5}{n^2} \leq \xi^2 \leq 1 - \frac{c_4}{n^2}, \\ n^{-\alpha-1}, & \xi^2 \leq \frac{c_5}{n^2} \end{cases}$$

where  $c_4, c_5 \in (0, 1)$  are arbitrary fixed numbers and concerns  $n$ .

**Lemma 2.** *Let  $w_\alpha(x)$  be the function  $w_\alpha(x) = |x|^{\frac{\alpha}{2}} \exp(-|x|^m)$  where  $\alpha/2 > -1$ ,  $m > 0$ . Then*

$$(5) \quad \lambda_n(w_\alpha, x) \leq K \exp(-|x|^m) \cdot \begin{cases} |x|^{\frac{\alpha}{2}} \cdot n^{\frac{1}{m}-1}, & c_7 n^{\frac{1}{m}-1} \leq |x| \leq c_6 n^{\frac{1}{m}}, \\ n^{(\frac{\alpha}{2}+1)(\frac{1}{m}-1)}, & |x| \leq c_7 n^{\frac{1}{m}-1} \end{cases}$$

further

a) in the case of  $m > 1$ :

$$\lambda_n(w_\alpha, x) \geq K \exp(-|x|^m) \cdot \begin{cases} |x|^{\frac{\alpha}{2}} \cdot n^{\frac{1}{m}-1}, & c_7 n^{\frac{1}{m}-1} \leq |x| \leq c_6 n^{\frac{1}{m}}, \\ n^{(\frac{\alpha}{2}+1)(\frac{1}{m}-1)}, & |x| \leq c_7 n^{\frac{1}{m}-1} \end{cases}$$

b) in case  $m = 1$  we have

$$\lambda_n(w_\alpha, x) \geq K \exp(-|x|) \cdot \begin{cases} \frac{|x|^{\frac{\alpha}{2}}}{\log(n+1)}, & \frac{c_8}{\log(n+1)} \leq |x| \leq c_6 n, \\ [\log(n+1)]^{-\frac{\alpha}{2}-1}, & |x| \leq \frac{c_8}{\log(n+1)}. \end{cases}$$

c) finally, in case  $0 < m < 1$ :

$$\lambda_n(w_\alpha, x) \geq K \exp(-|x|^m) \cdot \begin{cases} \frac{|x|^{\frac{\alpha}{2}+1-m}}{\log(n+1)}, & \frac{c_9}{(\log(n+1))^{\frac{1}{m}}} \leq |x| \leq c_6 n^{\frac{1}{m}}, \\ [\log(n+1)]^{-\frac{\alpha}{2m}-\frac{1}{m}}, & |x| \leq \frac{c_9}{(\log(n+1))^{\frac{1}{m}}}. \end{cases}$$

PROOF. 1. First we prove the upper estimate. We need a suitable polynomial  $P_{[n/2]}(x) \in \Pi_{[n/2]}$  which satisfies the following condition:

$$(6) \quad 0 < c_3 \leq P_{[n/2]}^2(x) \exp(-|x|^m) \leq c_3, \quad |x| \leq c_5 n^{\frac{1}{m}}.$$

We will obtain the desired polynomial using Lubinsky's function  $G$  defined by

$$(7) \quad G(x) = 1 + \sum_{k=1}^{\infty} \left( \frac{em}{2k} \right)^{\frac{2k}{m}} \cdot \frac{1}{\sqrt{k}} \cdot x^{2k}$$

([4], (17)) which originates from a function introduced by Mittag-Leffler cf. [5].

According to [4] Theorem 6 we have  $G(x) \asymp \exp(|x|^m)$ ,  $x \in \mathbb{R}$ . If we choose  $r_n$  to be the  $[n/4]$ -th partial sum of the power series in (7) then

$$(8) \quad 0 < r_n(x) \leq c \exp(|x|^m), \quad x \in \mathbb{R}.$$

Moreover, examination of the remainder term  $G - r_n$  shows that there exists  $b_0 > 0$  (absolute constant) such that  $G(x) \leq r_n(x) + o(1)$  uniformly for  $|x| \leq b_0 n^{1/m}$ ,  $n = 1, 2, \dots$  where  $\lim_{n \rightarrow \infty} o(1) = 0$ . Hence we have

$$(9) \quad \exp(|x|^m) \leq c r_n(x), \quad |x| \leq b_0 n^{1/m}.$$

Let  $P_{[n/2]}(x)$  be

$$P_{[n/2]}(x) = r_n \left( \frac{x}{\sqrt[m]{2}} \right).$$

Then from (8) and (9) we get

$$0 < c \exp(|x|^m) \leq P_{[n/2]}^2(x) \leq c \exp(|x|^m), \quad |x| \leq b_0 \sqrt[m]{2} n^{\frac{1}{m}},$$

i.e.  $P_{[n/2]}$  satisfies (6). It is well known that

$$(10) \quad \lambda_n(w_\alpha, x) = \min_{\substack{T_{n-1} \in \Pi_{n-1} \\ T_{n-1}(x)=1}} \int_{-\infty}^{\infty} |T_{n-1}(t)|^2 w_\alpha(t) dt.$$

Applying [6], Theorem 4.16.2, we obtain

$$(11) \quad \lambda_n(w_\alpha, x) \leq \min_{\substack{T_{[n/2]} \in \Pi_{[n/2]} \\ T_{[n/2]}(x)=1}} \int_{-c_1 n^{1/m}}^{-c_1 n^{1/m}} |T_{[n/2]}(t)|^2 \cdot |P_{[n/2]}(t)/P_{[n/2]}(x)|^2 w_\alpha(t) dt.$$

Thus by (6) we have

$$(12) \quad \frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \leq c \min_{\substack{T_{[n-2]} \in \Pi_{[n/2]} \\ T_{[n/2]}(x)=1}} \int_{-c_1 n^{1/m}}^{c_1 n^{1/m}} |T_{[n/2]}(t)|^2 \cdot |t|^{\frac{\alpha}{2}} dt,$$

where  $|x| \leq b_0 \sqrt[m]{2} \cdot n^{1/m}$ .

By a change of variables  $s = \frac{t}{c_1 n^{1/m}}$  we obtain from (12)

$$(13) \quad \frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \leq c n^{\frac{1}{m} + \frac{\alpha}{2} \cdot \frac{1}{m}} \cdot \lambda_{[n/2]+1} \left( v^{(\alpha/2)}, \frac{x}{c_1 n^{1/m}} \right),$$

$$|x| \leq b_0 \sqrt[m]{2} n^{\frac{1}{m}}.$$

From (13) the desired (5) follows.

2. Now we prove the lower estimate.

a) The case  $m > 1$ . There exists a polynomial  $P_n(x)$  of degree at most  $n$  for which

$$P_n^2 \asymp \exp(-|x|^m), \quad |x| \leq B n^{\frac{1}{m}},$$

([7], Theorem 1). Using these polynomials we can prove the estimates.

From (10)

$$\begin{aligned}
\lambda_n(w_\alpha, x) &\geq k \min_{\substack{T_{n-1} \in \Pi_{n-1} \\ T_{n-1}(x)=1}} \int_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)|^2 w_\alpha(t) dt \\
&\geq k \exp(|x|^m) \cdot \min_{\substack{T_{n-1} \in \Pi_{n-1} \\ T_{n-1}(x)=1}} \int_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)|^2 \cdot |P_n(t)/P_n(x)|^2 \cdot |t|^{\frac{\alpha}{2}} dt \\
&\geq k \exp(|x|^m) \cdot \min_{\substack{T_{2n-1} \in \Pi_{2n-1} \\ T_{2n-1}(x)=1}} \int_{-Bn^{1/m}}^{Bn^{1/m}} |T_{2n-1}(t)|^2 \cdot |t|^{\frac{\alpha}{2}} dt
\end{aligned}$$

and so

$$(14) \quad \frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \geq kn^{\frac{1}{m} + \frac{\alpha}{2} \cdot \frac{1}{m}} \cdot \lambda_{2n} \left( v^{(\alpha/2)}, \frac{x}{Bn^{1/m}} \right), \quad |x| \leq Bn^{\frac{1}{m}}.$$

Now we prove that Lemma 2, (a) is valid for  $|x| \leq cn^{1/m}$  where  $c$  is an arbitrary large constant. Since  $\lambda_n(w_\alpha, x)$  is a decreasing function of  $n$  we have

$$\frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \geq \frac{\lambda_{kn}(w_\alpha, x)}{\exp(-|x|^m)} \geq kn^{\frac{1}{m} + \frac{\alpha}{2} \cdot \frac{1}{m}} \cdot \lambda_{2kn} \left( v^{(\alpha/2)}, \frac{x}{B(kn)^{1/m}} \right),$$

$|x| \leq B(kn)^{1/m}$ , where  $k$  is arbitrary fixed integer. According to (4)

$$\lambda_{2kn} \left( v^{(\alpha)}, \xi \right) \asymp \lambda_n \left( v^{(\alpha)}, \xi \right).$$

Consequently Lemma 2 (a) is valid for  $|x| \leq cn^{1/m}$ .

b) Case  $m = 1$ . The calculation is the same as in a) but we use  $P_{Ln[\log(n+1)]}(x)$  instead of  $P_n(x)$ . Here  $P_{Ln[\log(n+1)]}(x)$  is of degree at most  $Ln[\log(n+1)]$  and satisfies

$$(15) \quad P_{Ln[\log(n+1)]}^2(x) \asymp \exp(-|x|), \quad |x| \leq Bn,$$

see [7].

Hence we get

$$(16) \quad \frac{\lambda_n(w_\alpha, x)}{\exp(-|x|)} \geq k \cdot n^{1 + \frac{\alpha}{2}} \cdot \lambda_{cn \log(n+1)} \left( v^{(\alpha/2)}, \frac{x}{Bn} \right), \quad |x| \leq Bn,$$

further

$$\frac{\lambda n(w_\alpha, x)}{\exp(-|x|)} \geq \frac{\lambda_{kn}(w_\alpha, x)}{\exp(-|x|)} \geq kn^{1+\frac{\alpha}{2}} \cdot \lambda_{cn \log(n+1)} \left( v^{(\alpha/2)}, \frac{x}{Bkn} \right),$$

$$|x| \leq Bkn,$$

where  $k$  is arbitrary fixed positive integer. Consequently Lemma 2 (b) is valid for  $|x| \leq cn$ , where  $c$  is an arbitrary large constant.

c) Case  $0 < m < 1$ . In this case we have

$$\begin{aligned} \lambda_n(w_\alpha, x) &\geq \min_{\substack{T_{n-1} \in \Pi_{n-1} \\ T_{n-1}(x)=1}} \int_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)|^2 w_\alpha(t) dt \\ &= k \min_{\substack{T_{n-1} \in \Pi_{n-1} \\ T_{n-1}(x)=1}} \int_{-B_1 n}^{B_1 n} |T_{n-1}(|s|^{1/m})|^2 \cdot |s|^{\frac{\alpha}{2m} + \frac{1}{m} - 1} \cdot e^{-|s|} ds \\ &\geq k \min_{\substack{T_{n/m} \in \Pi_{\frac{n}{m}} \\ T_{n/m}(|x|^m)=1}} \int_{-B_1 n}^{B_1 n} |T_{n/m}(s)|^2 \cdot |s|^{\frac{\alpha}{2m} + \frac{1}{m} - 1} \cdot e^{-|s|} ds \\ &= k \lambda_{n/m} \left( w_{\frac{\alpha}{m} + \frac{2}{m} - 2}, |x|^m \right). \end{aligned}$$

Here we can use the result of Lemma 2, (b) and thus we get Lemma 2, (c).

PROOF of the sharpness of Nikolskiĭ-type inequality.

Case (i):  $p \leq q$ .

We will obtain the polynomials  $R_n^*$  from D. S. LUBINSKY's function  $G$  defined by (7). According to [4] Theorem 6  $G(x) \asymp \exp(|x|^m)$ ,  $x \in \mathbb{R}$ , and thus if we define  $R_n^*$  to be the  $n/2$ -th partial sum of the power series in (17) then

$$(17) \quad 0 < R_n^*(x) \leq K \exp(|x|^m), \quad x \in \mathbb{R}.$$

Moreover, a close inspection of the remainder term  $G - R_n^*$  shows that there exists  $\varepsilon > 0$  such that  $G(x) \leq R_n^*(x) + o(1)$  uniformly for  $|x| \leq \varepsilon n^{1/m}$  and  $n = 1, 2, \dots$  where  $\lim_{n \rightarrow \infty} o(1) = 0$ . Hence we have

$$(18) \quad \exp(|x|^m) \leq K \cdot R_n^*(x), \quad |x| \leq \varepsilon n^{\frac{1}{m}}.$$

Let  $r > 0$ . Then we obtain

$$\|R_n^* w_\alpha\|_r \asymp \left\| R_n^* w_\alpha \chi_{[-cn^{1/m}, cn^{1/m}]} \right\|_r \asymp \left\| \chi_{[-cn^{1/m}, cn^{1/m}]} \cdot v_\alpha \right\|_r \asymp n^{\frac{\alpha}{2m} + \frac{1}{mr}},$$

where  $v_\alpha(x) = |x|^{\alpha/2}$ ,  $n = 1, 2, \dots$  which proves (3) in case  $p \leq q$ .

Case (ii):  $p > q$  and  $m > 1$ . Pick  $r_0 > 0$ . We will prove (3) by constructing a sequence of polynomials  $\{R_n^*\}_{n=1}^\infty$ ,  $\deg R_n^* \leq n$ , such that for every fixed  $r > r_0$

$$(19) \quad \|R_n^* \cdot w_\alpha\|_r \asymp n^{((1/m)-1)r}, \quad n = 1, 2, \dots$$

Given  $r_0 > 0$ , let us choose an integer  $M > 0$  such that  $r_0 > M^{-1}$ . Define the weight function  $u$  as follows

$$u(x) = |x|^{\frac{\alpha}{2M}} \cdot \exp(-|x|^m/M), \quad x \in \mathbb{R},$$

and let  $\{p_n(u, x)\}_{n=0}^\infty$  be the system of polynomials which is orthonormal with respect to  $u$ . Define the function  $K_n(u)$  by the formula

$$K_n(u, x, y) = \sum_{k=0}^{n-1} p_k(u, x) \cdot p_k(u, y).$$

and let  $N = \max \left\{ 1, \left\lfloor \frac{n}{2M} \right\rfloor \right\}$ . Then

$$(20) \quad R_n^* := \left( \frac{K_N(u, x, x_0)}{K_N(u, x_0, x_0)} \right)^{2M}$$

is a polynomial of degree at most  $n$ , where  $x_0 \neq 0$  is arbitrary fixed, independent of  $n$ . We may assume that  $x_0 > 0$ .

In what follows we will show that  $\{R_n^*\}_{n=1}^\infty$  satisfies (19). Using orthogonality it follows

$$\|R_n^* w_\alpha\|_{1/M} = \left\{ \int_R \frac{K_N^2(u, x, x_0)}{K_N^2(u, x_0, x_0)} u(x) dx \right\}^M = K_N^{-M}(u, x_0, x_0).$$

According to the Lemma 2, (a) we obtain  $K_N^{-1}(u, x_0, x_0) = O(N^{(1/m)-1})$ . Therefore we obtain

$$\|R_n^* w_\alpha\|_{1/M} \leq K n^{((1/m)-1)M}$$

and by (2)

$$(21) \quad \|R_n^* w_\alpha\|_r \leq K n^{(1-\frac{1}{m})(M-\frac{1}{r})} \cdot \|R_n^* w_\alpha\|_{1/M} \leq K \cdot n^{((1/m)-1)r}$$

holds for  $n = 1, 2, \dots$ . The next step is to show that there exists  $0 < \varepsilon < 1$  such that

$$(22) \quad R_n^*(x) \geq \frac{1}{2}, \quad |x - x_0| \leq \varepsilon n^{(1/m)-1}.$$

By (20)  $(R_n^*)^{\frac{1}{2M}}$  is a polynomial of degree at most  $n$  such that  $(R_n^*(x_0))^{\frac{1}{2M}} = 1$ . Hence for  $|x - x_0| \leq \varepsilon n^{(1/m)-1}$ ,  $n \geq n_0 = n_0(x_0, m)$ ,

$$\begin{aligned} \left| 1 - (R_n^*(x))^{\frac{1}{2M}} \right| &= \left| \int_x^{x_0} \left[ (R_n^*(t))^{1/2M} \right]' dt \right| \\ &\leq \sqrt{|x - x_0|} \cdot \left\{ \int_x^{x_0} \left| \left[ (R_n^*(t))^{\frac{1}{2M}} \right]' \right|^2 dt \right\}^{\frac{1}{2}} \\ &\leq k \sqrt{|x - x_0|} \left\{ \int_x^{x_0} \left| \left[ (R_n^*(t))^{\frac{1}{2M}} \right]' \cdot \sqrt{u(t)} \right|^2 dt \right\}^{\frac{1}{2}} \\ &\leq k \sqrt{|x - x_0|} \cdot \left\| \left[ (R_n^*(t))^{\frac{1}{2M}} \right]' \cdot \sqrt{u} \right\|_2. \end{aligned}$$

Here, using a Markov–Bernstein type inequality [1], the right-hand side can be estimated in terms of

$$\begin{aligned} &Kn^{(1-(1/m))} \sqrt{|x - x_0|} \cdot \left\| (R_n^*)^{\frac{1}{2M}} \sqrt{u} \right\|_2 \\ &= Kn^{(1-(1/m))} \cdot \sqrt{|x - x_0|} \cdot K_N^{-\frac{1}{2}}(u, x_0, x_0) = K \sqrt{|x - x_0|} \cdot n^{\frac{1-(1/m)}{2}}. \end{aligned}$$

Hence we get

$$\left| 1 - (R_n^*(x))^{\frac{1}{2M}} \right| = Kn^{\frac{1-(1/m)}{2}} \cdot \sqrt{|x - x_0|},$$

where  $|x - x_0| \leq \varepsilon n^{(1/m)-1}$ ; from this (22) immediately follows. From (22) we obtain

$$(23) \quad Kn^{((1/m)-1)r} \leq \|R_n^* w_\alpha\|_r, \quad n = 1, 2, \dots$$

From (21) and (23) the desired (19) follows.



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(Received March 9, 1994; revised July 10, 1995)