Some new algebraic equivalents of the Axiom of Choice

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The aim of this note is to prove the Theorem below. We shall work within ZF, i.e. ZERMELO—FRAENKEL *set theory without the Axiom of Choice*; we shall use the usual notations and terminology of set theory. In particular, ordinals are identified with their predecessors.

Theorem. *In ZF, the following statements are equivalent:*
(i) *Axiom of Choice.*
(ii) *On every nonempty set there exists a cancellative groupoid.*[1])

PROOF. First let us consider the implication (i)$\Rightarrow$(ii). In the case of a finite set (with n elements) choose the group of n-th roots of unity; in the case of a countably infinite set take, for instance, the additive group of the rational integers. In general, an algebraic structure $\langle X, +, 0\rangle$ is a group if and only if it is a model of a formula φ in a language L of first order predicate calculus containing the symbols $+$ and 0. (φ means the conjunction of the usual group axioms.) If a formula φ in a language with countably many symbols has an infinite model then, according to the upward LÖWEHEIM—SKOLEM theorem, it has a model of an arbitrary infinite power.[2])

As to the implication (ii)$\Rightarrow$(i): Evidently, it is sufficient to prove from (ii) that every non-empty set can be mapped on a suitable well-ordered set in a one-to-one way. We shall use the following

Lemma. (HARTOGS [1].) *In ZF, the following statement is true: Let A be an arbitrary set. Then there exists an ordinal α such that A does not possess any subset which can be mapped on α in a one-to-one way.*

PROOF OF THE LEMMA. Put

$$\alpha = \bigcup \{\text{type}\,(\langle X, R\rangle)+1 : X \subseteqq A, \quad R \subseteqq A\times A, \quad \text{and } R \text{ wellorders } X\},$$

where type $(\langle X, R\rangle)$ denotes the order type of $\langle X, R\rangle$. The right-hand side of this formula does indeed define a set. This follows from the conjunction of several axioms among them the Power Set Axiom and instances of the Replacement Scheme. Moreover, it is clear that α is an ordinal satisfying the requirements of the lemma.[3])

[1]) We remark that the property "cancellative" in both parts is essential and neither of them can be spared: Evidently, every set S becomes a right (but not left) cancellative semigroup be defining $x+y = x$ $(x, y \in S)$.

[2]) For other more algebraic proofs see e. g. [2].

[3]) This short proof was pointed out to us by A. MÁTÉ.

Let A be an arbitrary set and α an ordinal with the property described in the lemma. Let $\langle B, \prec \rangle$ denote a well-ordered set of type α. We may suppose $A \cap B = \emptyset$. Put $C = A \cup B$. According to (ii) there exists a binary operation $+$ on C such that $\langle C, + \rangle$ is a cancellative groupoid.

Now we shall prove the following proposition:

($*$) *For each $x \in A$ there is a $y \in B$ for which $x+y \in B$ holds.*
Suppose, ($*$) does not hold. Then there exists an $x \in A$ such that $x+y \in A$ for each $y \in B$. Because of the cancellation law, the function $f(y) = x+y$ ($y \in B$) maps the well-ordered set B of type α into A in a one-to-one way, which contradicts the choice of α. This proves ($*$).

Now, let be $D = B \times B$ and $\prec'$ the lexicographic ordering, which is a well-ordering of D obtained from $\prec$. We define a map $g: A \to D$ as follows:

For $x \in A$ put $g(x) = \min_{\prec'} \{\langle y, z \rangle \mid y, z \in B \text{ and } x+y = z\}$. The ordering $\prec'$ being a wellordering, by ($*$), the mapping g is defined for each $x \in A$ and is single-valued. Furthermore, g is one-to-one, because, in view of the cancellation law, from $x \neq x'$ it follows that $g(x) \neq g(x')$. Hence g is a one-to-one mapping of A onto a subset of the well-ordered set D. Consequently, the set A can be well-ordered.[4])

Corollary. *The Axiom of Choice is equivalent to any of the assertions listed under* ($**$).

($**$) *On every nonempty set there exists,*

(a) *a cancellative semigroup,*

(b) *a cancellative abelian semigroup,*

(c) *a quasigroup,*

(d) *a loop,*

(e) *a group,*

(f) *an abelian group,*

(g) *a ring,*

(h) *a commutative ring,*

(i) *an integral domain with unit.*

References

[1] F. Hartogs, Über das Problem der Wahlordnung, *Math. Ann.* **76** (1915) 438—443.

[2] A. Kertész, Das Kuratowski-Zornsche Lemma und seine Anwendungen in der Algebra, Lecture Notes, University of Jyväskylä, *Jyväskylä* 1973.

[3] A. Tajtelbaum—A. Tarski, Sur quelques théorèmes qui équivalent à l'axiome du choix. *Fund. Math.* **5** (1924), 197—201.

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[4]) The proof of the implications $(ii) \Rightarrow (i)$ is an unessential modification of a proof of the following theorem due to A. Tarski: If $m^2 = m$ holds for every infinite power m, then the Axiom of Choice also holds [3].