Regular congruence on a simple semigroup with a minimal right ideal

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In the first part of this paper, two characterizations of a group congruence on a simple semigroup with a minimal right ideal are given (1. 3), (1. 9). The conditions given in (1. 3) generalize Schwarz's conditions for a group congruence on a completely simple semigroup (1. 4).

In the second section, it is shown that any regular[1] congruence on a right simple semigroup is characterized as the intersection of a group congruence and a band congruence (2. 5). It is also shown that every simple semigroup with a minimal right ideal has a minimum completely simple congruence (2. 7).

Since all regular congruences on simple semigroups with a right ideal are completely simple congruences, it follows that the lattice of regular congruences for such semigroups is isomorphic to the lattice of congruences on a completely simple semigroup (2. 8). This lattice is described by KAPP and SCHNEIDER [3].

The basic properties of simple semigroups with a minimal right ideal may be found in section 8. 2 [1]. The terminology and notation will be that of CLIFFORD and PRESTON [1].

1. Group congruences

It is clear that a right simple semigroup is simple with a minimal right ideal. For the convenience of the reader, we will state without proofs TEISSIER's characterization of a group congruence [6] on such a semigroup.

Recall the following:

(1. 1) *Definition.* ([1], 55, Vol. II.) A subset U of a semigroup S is said to be *left [right] unitary in* S if $u \in U$ and $ux \in U$ $[xu \in U]$, for $x \in S$, together imply that $x \in U$. A subset U which is both right and left unitary in S is said to be *unitary in* S.

(1. 2) **Theorem.** ([6]). *A right simple semigroup, S, without idempotent elements has a group congruence ϱ if and only if S contains a subsemigroup, E, which is unitary in S and satisfies $Eae \subseteq aE$ for all $ae \in SE$. Moreover, E is the kernel of ϱ and $a\varrho b$ for all $a, b \in S$, if and only if $aE = bE$.*

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[1]) Let P be an abstract property of semigroups. Then a congruence ϱ of the semigroup S has the property P if S/ϱ has this property. E. g. P con be: to be group, regular semigroup.

One may use techniques similar to those used by Teissier to generalize (1. 2).

(1. 3) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then S has a group congruence, ϱ, if and only if S contains a unitary subsemigroup, E, such that for all $x, y \in S$, $xEy \subseteqq ExyE$. Moreover, E is the kernel of ϱ, and $(a, b) \in \varrho$ if and only if $EaE = EbE$.*

We note here that in (1. 3), unlike (1. 2), S is not required to be idempotent free, and that a simple semigroup with a minimal right ideal is completely simple if and only if its contains an idempotent ([1] Theorem 8. 14). Thus as a corollary to (1. 3), we have:

(1. 4) *Corollary* ([5] (4) Theorem.) Let S be a completely simple semigroup. Then S has a group congruence if and only if S contains a simple semigroup, E, which contains all of the idempotents of S, and for which there exists at least one **H**-class, H_a of S such that $E \cap H_a$ is a normal subgroup of H_a. In this case, E is the kernel of the congruence, and the congruence classes are of the form ExE for $x \in S$.

We now give another characterization of a group congruence on a simple semigroup with a minimal righ ideal.

(1. 5) **Lemma.** ([1] Lemma 8. 13.) *Let S be a simple semigroup with a minimal right ideal. Then S is the disjoint union of its minimal right ideals; xS is the minimal right ideal containing x; every minimal right ideal is a right simple semigroup.*

(1. 6) *Definition.* Let E be a subset of a semigroup T. We say E is a *right cross-section* if $E \cap R_a \neq \square$, for all $a \in T$. $\square$ denoting the empty set.

(1. 7) *Note.* It is easily checked that if E is a class of a congruence ϱ on a semigroup T, then if $x, y \in EaE$ for any $a \in T$, $(x, y) \in \varrho$.

(1. 8) **Lemma.** *Let ϱ be a congruence on S. Let E be a ϱ-class which is also a unitary subsemigroup of S and satisfies:*

i) *E is a right cross-section.*

ii) *For any $a \in S$, there exists $e \in E$ such that $a = ae$.*

Then for any ϱ-class, U, there exists $A \subseteqq S$ such that $U = \bigcup_{x \in A} ExE$.

Proof. Let $s \in U$, then by i), $E \cap sS \neq \square$. By (1. 5), sS is the minimal right ideal of S containing s, and hence if $e \in E \cap sS$, $eS = sS$. But then there exists $s_1 \in S$, such that $s = es_1$. By ii), there exists $e_1 \in E$, such that $s_1 = s_1 e_1$. We combine these equations to get $s = es_1 e_1 \in Es_1 E$, so that every element of U is contained in a set ExE for some $x \in S$, and there exists a subset A of S, such that $U \subseteqq \bigcup_{x \in A} ExE$. But from (1. 7), it is clear that if $(ExE) \cap U \neq \square$ for any $x \in S$, then $ExE \subseteqq U$. It now follows that there exists a subset A of S such that $U = \bigcup_{x \in A} ExE$.

The second characterization follows:

(1. 9) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then if S has a group congruence, ϱ, E, the kernel of ϱ is a unitary subsemigroup such that:*

i) *E is a right cross-section.*

ii) *For $a \in S$, $a = ae$ for some $e \in E$.*

iii) *If $xEy \cap E \neq \square$, for $x, y \in S$, then $xEy \subseteqq E$.*

Conversely, if E is a unitary subsemigroup of S satisfying i)—iii), *then there exists a group congruence, ϱ, with E as its kernel.*

Proof. Suppose that S has a group congruence, ϱ. Then by (1. 3), the kernel of ϱ, E, is a unitary subsemigroup of S, such that for all $x, y \in S$, $xEy \subseteqq ExyE$. The **R**-classes of S are of the form aS where $a \in S$ (1. 5), so that E is a right cross-section if for every $a \in S$, $E \cap aS \neq \square$. But the collection of sets EaE forms a group, (1. 3) and each element EaE has an inverse EbE which satisfies $(EaE)(EbE) = EabE = E$. It is easily shown that this implies $ab \in E$, but $ab \in aS$, hence E is a right cross-section. If $aEb \cap E \neq \square$, we apply (1. 3) to get $EabE \cap E \supseteqq aEb \cap E \neq \square$. But then by (1. 7), it follows that $EabE = E$. Thus $aEb \subseteqq EabE = E$. Therefore i) and iii) hold. We now show that ii) is valid. Let $a \in S$. We know that aS is the minimal right ideal containing a (1. 5) and therefore $a = as$ for some $s \in S$. By (1. 3), the ϱ-classes of S are of the form ExE, and we have $(EaE)(EsE) = EasE = EaE$. Since S/ϱ is a group, EsE must be the group identity, E. Thus $s \in E$ for E is unitary, and we have ii).

Conversely, suppose that S has a unitary subsemigroup, E, satisfying i)—iii). Teissier ([7], 2) has shown that condition iii) is a necessary and sufficient condition for the existence of a congruence, ϱ, for which E is a ϱ-class. For every ϱ-class, U, of S there exists a subset A of S, such that $U = \bigcup_{x \in A} ExE$, by (1. 8). Clearly $E^2 \subseteqq E$, so that E is a right identity for S/ϱ. To show S/ϱ is a group, we need only show that every ϱ-class, U, has a right inverse. Let $a \in S$ for which $EaE \subseteqq U$. By i), $E \cap aS \neq \square$, and there is an $s \in S$ such that $as \in E$. Let U' be the ϱ-class containing EsE, then $UU' \supseteqq (EaE)(EsE)$. By ii), there exists $e \in E$ such that $a = ae$. Then we have $aes = as \in aEs \cap E$, and by iii), $aEs \subseteqq E$. Thus $UU' = E$, and U has a right inverse. Therefore S/ϱ has a right identity and every element of S/ϱ has a right inverse, hence, S/ϱ is a group.

The following is a simple example of a non-trivial group homomorphism on a simple semigroup with more than one minimal right ideal and without idempotents. For a more complex example, see [4].

(1. 10) *Example*. Let $S = \Lambda \times B$ the direct product of a nontrivial left zero semigroup, Λ, and a Baer—Levi semigroup, B, of type (p, q) with $p > q$ ([1] § 8. 1). Since B is right simple idempotent free, S is simple, and $\lambda \times B$ is a minimal right ideal for each $\lambda \in \Lambda$. Clearly S has no idempotent. Define the map $\alpha : S \to B$ as follows: $(\lambda, a)\alpha = a$, for all $(\lambda, a) \in S$. It is easily shown that α is a homomorphism of S onto B. One can use (1. 2) to show that there is a non-trivial homomorphism, δ, of B onto a group [4]. Clearly $\alpha\delta$ is a non-trivial homomorphism of S onto a group.

The following lemma is easily proven.

(1. 11) **Lemma.** *Let S be a semigroup with subsemigroup T, and let ϱ be a congruence on S. Then $\varrho/T = (x, y) : x, y \in T, (x, y) \in \varrho\}$ is a congruence on T, and if $a(\varrho/T)$ is the ϱ/T-class of $a \in T$, then $a(\varrho/T) = T \cap a\varrho$, where $a\varrho$ is the ϱ-class of a.*

(1. 12) *Proposition*. Let S be a simple semigroup with a minimal right ideal. If τ is a group congruence on S, then for any $a \in S$, $aS/(\tau/aS)$ is isomorphic to S/τ.

Proof. Let E be the kernel of τ. It is easily checked that $ExE \cap aS \neq \square$ for any $x \in S$, since by (1. 9), E is a right cross-section. From this it follows that for every $x \in S$, there a $y \in aS$, for which $ExE = EyE$. Thus if for every $x \in aS$ we define $(ExE)\theta = ExE \cap aS$, one can easily ckeck that θ is an isomorphism of S/τ onto $(aS)/(\tau/aS)$.

2. The lattice of regular congruences on simple semigroups with a minimal right ideal

We recall that in general, a semigroup need not have a minimum group congruence. However, we now show that every right simple semigroup has a minimum group congruence.

(2. 1) *Proposition.* Let S be a right simple semigroup, then S has a minimum group congruence.

PROOF. One can easily show that S has a minimum cancellative congruence, μ, by ([1], Theorem 1. 7). S/μ is right simple and left cancellative, hence a right group ([1] 1. 1). Thus S/μ has an idempotent, but it is also right cancellative, therefore S/μ is a group ([1] vol. II, 85, ex. 5). If σ is any group congruence, then σ is a cancellative congruence, and $\mu \subseteq \sigma$. Thus μ is the minimum group congruence on S.

We recall that the homorphic image of a simple semigroup with a minimal right ideal is a simple semigroup with a minimal right ideal. Hence, if the homorphic image of such a semigroup has an idempotent, it is a completely simple semigroup ([1] Theorem 8. 14). Thus every regular congruence on a simple semigroup with a minimal right ideal is a completely simple congruence.

We will now find a minimum completely simple congruence.

(2. 2) *Notation.* Let S be a simple semigroup with a minimal right ideal. Then for every $a \in S$, aS is a right simple semigroup (1. 5), so that aS has a minimum group congruence, which we denote by γ_a, and a minimum band congruence, which we denote by β_a.

We now quote a theorem of HOWIE and LALLEMENT [2] which will form the basis for the main result.

(2. 3) **Theorem.** ([2] Theorem 4. 1.) *Let S be a regular semigroup. If τ is a group congruence on S, and if σ is a band congruence on S, then $S/(\tau \cap \sigma)$ is a band of groups whose idempotents form a unitary subsemigroup. Conversely, if ϱ is a congruence on S and S/ϱ is a band of groups whose idempotents form a unitary subsemigroup, then $\varrho = \tau \cap \sigma$ where τ is a group congruence on S and σ is a band congruence on S. Moreover, τ and σ are uniquely determined by ϱ.*

(2. 4) **Lemma.** *Let S be a right simple semigroup. If τ is a group congruence on S, and if σ is a band congruence on S, then $S/(\tau \cap \sigma)$ is regular.*

PROOF. If E is the kernel of τ, and $x \in E$, then $(x, x^2) \in \tau$. Clearly $(x, x^2) \in \sigma$, therefore $(x, x^2) \in \tau \cap \sigma$. Thus $S/(\tau \cap \sigma)$ is right simple with an idempotent, hence it is regular.

Next we generalize (2. 3) to right simple semigroups.

(2. 5) **Theorem.** *Let S be a right simple semigroup. If τ is a group congruence on S, and if σ is a band congruence on S, then $S/(\tau \cap \sigma)$ is regular. Moreover, if σ is a regular congruence on S, then $\varrho = \tau \cap \sigma$ where τ is a group congruence on S and σ is a band congruence on S. In this case, τ and σ are uniquely determined by ϱ.*

PROOF. $\tau \cap \sigma$ is a regular congruence by (2. 4).

Let ϱ be a regular congruence on S. Then S/ϱ is a right group ([1] Theorem 1. 27), hence isomorphic to $G \times E$ where G is a group and E is a right zero semigroup ([1] Theorem 1. 27). One easily checks that the idempotents of $G \times E$ are of the form (e, μ) where e is the identity of G and μ is an arbitrary element of E. It is clear from this that the idempotents of $G \times E$, and hence of S/ϱ, form a unitary subsemigroup. Since S/ϱ is a regular semigroup, we may now apply (2. 3) to $\Delta_{S/\varrho}$, the identity relation on S/ϱ, to get $\Delta_{S/\varrho} = \tau' \cap \sigma'$ where $\tau'[\sigma']$ is a group [band] congruence on S/ϱ. By ([1], Theorem 1. 5), there exists $\tau[\sigma]$ a group [band] congruence on S containing ϱ such that $\tau' = \tau/\varrho[\sigma' = \sigma/\varrho]$. Then $(\tau/\varrho \cap (\sigma/\varrho) = \Delta_{S/\varrho}$ and thus $\varrho = \tau \cap \sigma$. is regular, therefore τ' and σ' are uniquely determined by (2. 4), and hence τ and σ are uniquely determined by ϱ.

(2. 6) **Lemma.** *Let S be a simple semigroup with a minimal right ideal, and π be the congruence generated by the relation $\alpha = \bigcup_{x \in A} (\gamma_a \cap \beta_a)$. Then S/π is a completely simple semigroup.*

PROOF. Clearly S/π is simple with a minimal right ideal. Let

$$\pi/aS = \{(x, y) : x, y \in S,\ (x, y) \in \pi\}$$

for $a \in S$. By (1. 11), π/aS is a congruence on aS, and $\gamma_a \cap \beta_a \subseteqq \pi/aS$. Then $aS/(\pi/aS)$ is regular, since by ([1] Corollary 1. 62), it is the homomorphic image of $aS/(\gamma_a \cap \beta_a)$ which is regular (2. 3). Let $e(\pi/aS)$ be an idempotent element of $aS/(\pi/aS)$, then $e^2 \in e^2(\pi/aS) = [e(\pi/aS)]^2 = e(\pi/aS)$. But we have noted in (1. 11) that $e(\pi/aS) = e\pi \cap aS$, therefore $e^2 \in e\pi$. Since $e^2 \in e^2\pi = [e\pi]^2$ and π is an equivalence relation, we have $[e\pi]^2 = e\pi$, so that $e\pi$ is an idempotent element of S/π. Thus S/π is simple with a minimal right ideal, and has an idempotent, therefore it is completely simple by ([1], Theorem 8. 14).

(2. 7) **Theorem.** *Let S be a simple semigroup with a minimal right ideal. Then π, as in (2. 6), is the minimum completely simple congruence on S.*

PROOF. Let ϱ be any completely simple congruence on S. For every $a \in S$, it is easily shown that the collection of all ϱ-classes of S which have non-trivial intersection with aS is an **R**-class of S/ϱ. So there exists $e \in aS$ for which $e\varrho = (e\varrho)^2$. But then, $e\varrho \cap aS = (e\varrho)^2 \cap aS$ and by (1. 11), we have ϱ/aS is a regular congruence on aS. We recall that all regular congruences on right simple semigroups can be written as the intersection of a group congruence and a band congruence (2. 5). Therefore $\varrho/aS = \tau \cap \sigma$, where τ is a group congruence on aS and σ is a band congruence on aS. Then $\varrho/aS = \tau \cap \sigma \supseteqq \gamma_a \cap \beta_a$. Since a is arbitrary, $\varrho \supseteqq \gamma_a \cap \beta_a$ for all $a \in S$. But π is the smallest congruence containing $\gamma_a \cap \beta_a$ for all $a \in S$, hence $\varrho \supseteqq \pi$. Thus π is the minimum completely simple congruence on S.

(2. 8) **Theorem.** *Let S be a simple semigroup with a minimal right ideal, and π be the minimum completely simple congruence on S. Then if ϱ is a regular congruence on S, $\pi \subseteqq \varrho$, and ϱ/π is a congruence on S/π. Let θ be the map of **C**, the lattice of regular congruences on S, to C', the lattice of congruences on S/π, defined by $\varrho\theta = \varrho/\pi$. Then θ is a lattice isomorphism of C onto C'.*

The lattice of congruences on a completely simple semigroup, hence on S/π, is discussed in detail in [3]. Two immediate consequences of (2. 8) are:

(2. 9) *Corollary*. The lattice of regular congruences on a simple semigroup with a minimal right ideal is semimodular.

(2. 10) *Corollary*. Let S be a simple semigroup with a minimal right ideal. Then S has a minimum group congruence.

We now generalize (2. 5).

(2. 11) *Corollary*. Let S be a simple semigroup with a minimal right ideal. Then ϱ is a regular congruence on S, for which the idempotents of S/ϱ form a unitary subsemigroup of S/ϱ if and only if $\varrho = \tau \cap \sigma$, where τ is a group congruence on S and σ is a band congruence on S. Moreover, τ and σ are uniquely determined by ϱ.

PROOF. If ϱ is a regular congruence with the idempotents of S/ϱ forming a unitary subsemigroup, then $\pi \subseteqq \varrho$. But then there exists ϱ', a congruence on S/π, such that $\varrho' = \varrho/\pi$. We know that S/π is regular, and hence by (2. 3), it follows that $\varrho' = \tau' \cap \sigma'$, where $\tau'[\sigma']$ is a group [band] congruence on S/π. In fact, τ' and σ' are uniquely determined by ϱ'. But then there exist congruences τ and σ on S containing π that satisfy $\tau' = \tau/\pi$ and $\sigma' = \sigma/\pi$, by ([1] Theorem 1. 6). Clearly $\tau[\sigma]$ is a group [band] congruence on S, and $\varrho = \tau \cap \sigma$. Moreover, since τ' and σ' are uniquely determined by ϱ', we have τ and σ are uniquely determined by ϱ.

Conversely, suppose that $\varrho = \tau \cap \sigma$, where $\tau[\sigma]$ is a group [band] congruence on S. Then we have $\pi \subseteqq \tau$ and $\pi \subseteqq \sigma$, so that by ([1] Theorem 1. 6) we have $\tau' = \tau/\pi$ $[\sigma' = \sigma/\pi]$ is defined and is a group [band] congruence on S/π. Clearly $(S/\pi)/(\tau' \cap \sigma')$ is isomorphic to S/ϱ. But S/π is regular, and applying (2. 3), we see that the idempotents of $(S/\pi)/(\tau' \cap \sigma')$ form a unitary subsemigroup. Thus the idempotents of S/ϱ form a unitary subsemigroup of S/ϱ, and we have our result.

We close by noting that since the semigroup, S, of example (1. 10) is a simple, idempotent-free semigroup, $\mathbf{R}$ is a band congruence on S ([1] ex. 1, 93, Vol. II). But S has a non-trivial group congruence, ϱ, therefore by (2. 11), $\varrho \cap \mathbf{R}$ is a non-trivial, completely simple congruence on S. For other examples, see [4].

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(Received July 8, 1971.)