

Abelian groups in which every neat subgroup is a direct summand

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1. Introduction

When investigating the structure of a given abelian group G it is often of value to know how certain subgroups sit in G . For example, KERTÉSZ [4] has shown that every subgroup of G is a direct summand if and only if every element in G has square-free order. FUCHS, KERTÉSZ and SZELE [3] have shown that

Theorem A. *An abelian group G has the property that all its pure subgroups are direct summands if and only if it has one of the following forms:*

- (1). *G is a torsion group, each p -primary component of which is the direct sum of a bounded and a divisible group.*
- (2). *$G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups.*

Neat subgroups are just those without proper essential extensions in the containing group and therefore, from a purely module-theoretic point of view, neatness is a more natural concept than purity. Thus we are led to classify those abelian groups having the property that every neat subgroup is a direct summand (Theorem 1). As an application of this result, we classify the quasi-injective abelian groups (Theorem 2).

2. Preliminaries

By a *group* we shall mean an additively written abelian group. If A is a subgroup of G , then A is *neat* in G if and only if $pG \cap A = pA$ for all primes p ; A is *pure* in G if and only if the equation $p^n G \cap A = p^n A$ holds for all primes p and all positive integers n . A group G has *Property N* if and only if every neat subgroup of G is a direct summand. In our discussions we will often use the fact that direct summands are pure.

For any group G , tG and G_p will denote the maximal torsion subgroup of G and the p -primary component of G , respectively. We use the symbol $+$ to denote direct sums. Occasionally we will refer to

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Lemma 1. *If G satisfies Property N and if H is a direct summand of G , then H also satisfies Property N .*

PROOF. Neatness of A in H implies neatness in G . Thus $G = A + B$ for some subgroup B of G and $H = A + (B \cap H)$.

3. Torsion groups with Property N

Proposition 1. *If a p -group G satisfies Property N , then either G is divisible or*
 (*) $G = H + K$ *where H is the direct sum of cyclic groups of order p^n and K is the direct sum of cyclic groups of order p^{n+1} .*

PROOF. By Theorem A, a p -group satisfying our property must have a decomposition $G = D + B$ where D is divisible and B is bounded. To see that one of D or B is zero, we assume the existence of a quasicyclic direct summand D' of D and a cyclic summand C of B having order p^n for some positive integer n . Let $G' = D' + C$. If $d \in D'$ is of order p^{n+1} and c is a generator of C , then the subgroup A , generated by (d, c) , is neat in G' but not pure in G' . Thus A is not a direct summand of G' and G' does not satisfy Property N . According to Lemma 1, G does not satisfy our property. This contradiction forces us to conclude that either D or B is zero, i.e., either G is bounded or G is divisible.

In the case G is bounded and B_1 and B_2 are cyclic summands of G having orders p^n and p^{n+m} , respectively, we assume that $m > 1$. Let b_1 generate B_1 and b_2 generate B_2 . Then the subgroup of $G' = B_1 + B_2$ generated by (b_1, pb_2) is neat but not pure in G' . Again, this yields a contradiction and we have forced to conclude that $m \leq 1$.

For the converse of Proposition 1 we need

Lemma 2. *If a p -group has a decomposition (*), then every neat subgroup is pure.*

PROOF. Let $\{H_i: i \in I\}$ denote the cyclic summands of H having order p^n and $\{K_j: j \in J\}$ those of K having order p^{n+1} . Let A be neat in G . Note that, since $p^k G$ is zero for all $k \geq n+1$, $p^k G \cap A = p^k A$ for all such k . Thus we now assume that $p^m G \cap A = p^m A$ for some m , $1 \leq m < n$. If $p^{m+1}x \in A$ for some x in G , then, by assumption, there is an $a \in A$ satisfying $px - a \in G[p^m]$. If $(a)_i$ and $(x)_i$ denote the i -th coordinates of a and x , respectively, then $p(x)_i - (a)_i \in H_i[p^m] = p^{n-m}H_i \subseteq pH_i$. Thus $(a)_i$ is contained in pH_i for each $i \in I$. Similarly, $(a)_j$ is contained in pK_j for each $j \in J$. Consequently, there is a $g \in G$ with $pg = a$. By the neatness of A , we are guaranteed an element $a' \in A$ satisfying $pg = a = pa'$. But then $p^{m+1}x = p^{m+1}a' \in p^{m+1}A$ and A is pure in G .

We now have the converse of Proposition 1 in

Proposition 2. *If a p -group is divisible or has a decomposition (*), then G satisfies Property N .*

PROOF. Since every neat subgroup of a divisible group is divisible, we conclude that our class contains the divisible groups. So suppose that G has a decomposition (*) and that A is neat in G . Then A is pure by the previous lemma and A is a direct summand since A is bounded (KULIKOV [2]).

We are now able to give the main result of this section.

Proposition 3. A torsion group G satisfies Property N if and only if each p -primary component is divisible or has a decomposition (*).

PROOF. The necessity follows from Lemma 1 and Proposition 1. For the sufficiency suppose that A is neat in G and that each G_p is divisible or enjoys a decomposition (*). As A_p is neat in G_p for each p , Proposition 2 implies $G_p = A_p + B_p$ for some subgroup B_p and thus $G = A + B$ where B is the direct sum of the complements B_p .

4. Arbitrary groups with Property N .

Proposition 4. If $G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups, then G satisfies Property N .

PROOF. Let A be neat in G . Since tA is divisible, $A = tA + B$ where B is torsion-free. As $tG \cap B$ is zero, we may argue in the usual way that $G = tG + H$ where B is a subgroup of H . Now B is neat, hence pure, in the torsion-free H . From Theorem A we conclude that $H = B + H'$ and as a consequence, $G = tG + H = (tA + G') + (B + H') = A + (G' + H')$; that is, G has Property N .

Theorem 1. An abelian group G has Property N if and only if G has one of the following forms:

(1'). G is a torsion group in which, for each prime p , G_p is either divisible or else of the form $H + K$ where H is a direct sum of cyclic groups of order p^n and K is a direct sum of cyclic groups of order p^{n+1} .

(2). $G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups.

PROOF. Let G satisfy Property N . If G is torsion, we apply Proposition 3 and find that G is of the form (1'). If, on the other hand, G is not torsion, then Theorem A implies that G has the form (2). (We have used the fact that if G has Property N , then pure subgroups of G are direct summands.) Proposition 3 and Proposition 4 yield the converse.

5. A characterization of quasi-injective abelian groups

We define an abelian group G to be *quasi-injective* if and only if each homomorphism of any subgroup A into G can be extended to a homomorphism of G into G . A subgroup A of G is *closed* in case A has no proper essential extensions in G . It is well known that A is neat if and only if A is closed. Thus the class of abelian groups whose neat subgroups split off contains the quasi-injectives. This follows from

Theorem B. (FAITH [1].) *If G is quasi-injective then every closed subgroup of G is a direct summand.*

Proposition 5. If G is quasi-injective, then either G is divisible or (**) G is torsion and each G_p is the direct sum of mutually isomorphic cocyclic groups.

PROOF. It suffices to apply Theorem 1 and Lemma 1 to G after showing that none of the following types of groups are quasi-injective.

- (a) $G = H + K$ where H is cyclic of order p^n and K is cyclic of order p^{n+1} .
- (b) G is reduced torsion-free group having rank one.

For the case (a), we let k and h generate K and H , respectively. If H' is the subgroup of G generated by pk , then we define a homomorphism from H' to G by $f(pk) = h$. It is easy to see that no extension of f could be defined at the generator k . Thus f does not extend.

In (b) we view G as a subgroup of the additive group of rationals Q and then suppose that $1/t \in Q - G$ while $0 \neq m/n \in G$. Letting A denote the subgroup of G generated by $m^2 t/n$ we map A into G by defining $f(m^2 t/n) = m/n$. Assume that f extends to an endomorphism F of G and that $F(m) = x \in G$. Then $f(m^2 t) = m$ while $F(m^2 t) = mtx$. We now conclude that $x = 1/t \in G$, contrary to assumption. Consequently, G is not quasi-injective.

Proposition 6. If G satisfies $(**)$ then G is quasi-injective.

PROOF. Let A be a subgroup of G and f a homomorphism from A into G . Then, for each prime p , we have induced homomorphisms $f_p: A_p \rightarrow G_p$. As G is the direct sum of its components G_p , extending each f_p to an endomorphism F_p of G_p is sufficient to extend f to an endomorphism of G . Now since divisible groups are quasi-injective, we need only consider the components of G which are direct sums of cyclic groups of order, say, p^n . Let D denote the minimal injective extension of G_p . Then D is a divisible p -group whose fully invariant subgroups are those of the form $D[p^k]$ for k a positive integer. Consequently G_p is fully invariant in D . The injectivity of D allows us to extend f_p to an endomorphism F of D . Setting F_p equal to the restriction of F to G_p yields the extension we need. Thus G is quasi-injective.

Proposition 5 and Proposition 6 give us

Theorem 2. G is quasi-injective if and only if G is divisible or G satisfies $(**)$.

As a final note we observe that those groups in which every subgroup is neat are precisely the elementary groups. This follows from the argument given in Theorem 4 of [3].

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