Laplace—Stieltjes transform of Mikusinski operator functions

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I.

Introduction

The purpose of this paper is to define and develop an integral which can be used to generalize the Laplace—Stieltjes transform. This generalization will apply to functions of a real variable which take their values in Mikusinski's [4] operator space.

In [1] and [2] GESZTELYI defines Stieltjes integrals of operator functions. In [1] he begins with the integral of f with respect to g where f is operator-valued and g is numerical-valued and in [2] he defines an integral as the limit of a summation. Each of these approaches seems indirect in trying to generalize the Laplace—Stieltjes transform; hence we will define an integral of f with respect to g where f is numerical-valued and g is operator-valued.

1. *The integral over a bounded interval*

Definition 1. A function $g(\lambda, t)$ is of class H on $[a, b]\times[0, \infty)$ if for each λ in $[a, b]$, $g(\lambda, t)$ is continuous in t on $[0, \infty)$ and for each $T>0$ there is an M such that if $0\leqq t_0\leqq T$ then the variation of $g(\lambda, t_0)$ on $[a, b]$ is less than M.

Theorem 1. *If $f(\lambda)$ is continuous on $[a, b]$ and $g(\lambda, t)$ is of class H on $[a, b]\times[0, \infty)$ then $h(t) = \int_a^b f(\lambda)d_\lambda g(\lambda, t)$ is continuous on $[0, \infty)$.*

PROOF. Follows from a theorem in Hildebrandt [3, p. 78].

Definition 2. Operator function g is of class H on $[a, b]$ if there exists an operator P such that $g(\lambda)=P\{q(\lambda, t)\}$ on $[a, b]$ and $q(\lambda, t)$ is of class H on $[a, b]\times[0, \infty)$. We say $P\{q(\lambda, t)\}$ is a representation for $g(\lambda)$ on $[a, b]$.

Definition 3. If $f(\lambda)$ is continuous on $[a, b]$ and $g(\lambda)$ is of class H on $[a, b]$ with representation $P\{q(\lambda, t)\}$ then

$$\int_a^b f(\lambda)dg(\lambda) = P\Big\{\int_a^b f(\lambda)d_\lambda q(\lambda, t)\Big\}$$

Theorem 2. *If $f(\lambda)$ is continuous and $g(\lambda)$ of class H on $[a, b]$ then $\int_a^b f(\lambda)dg(\lambda)$ exists and is unique.*

PROOF. Existence follows from Theorem 1. For uniqueness we let $g(\lambda)$ have representations $P_1\{q_1(\lambda, t)\}$ and $P_2\{q_2(\lambda, t)\}$ on $[a, b]$. Without loss of generality we assume P_1 and P_0 are of class C (i.e. continuous on $[0, \infty)$). For $i=1, 2$ we have $q_i(\lambda, t)$ is continuous in t for $t \geqq 0$ for each λ in $[a, b]$ and it is easily verified that for each $t>0$, $P_i(t-\mathscr{T})q_i(\lambda, \mathscr{T})$ is of uniformly bounded variation in λ over $[a, b]$ for $\mathscr{T}$ in $[0, t]$. Therefore by a theorem in WIDDER [5, p. 25] we have

$$\int_0^t P_i(t-\mathscr{T})d\mathscr{T}\int_a^b f(\lambda)d_\lambda q_i(\lambda,\mathscr{T}) = \int_a^b f(\lambda)d_\lambda \int_0^t P_i(t-\mathscr{T})q_i(\lambda,\mathscr{T})d\mathscr{T}, \quad \text{for} \quad i=1,2.$$

Therefore,

$$\begin{aligned} P_1\Big\{\int_a^b f(\lambda)d_\lambda q_1(\lambda,t)\Big\} &= \int_0^t P_1(t-\mathscr{T})\Big[\int_a^b f(\lambda)d_\lambda q_1(\lambda,\mathscr{T})\Big]d\mathscr{T} = \\ &= \int_a^b f(\lambda)d_\lambda \int_0^t P_1(t-\mathscr{T})q_1(\lambda,\mathscr{T})d\mathscr{T} = \\ &= \int_a^b f(\lambda)d_\lambda \int_0^t P_2(t-\mathscr{T})q_2(\lambda,\mathscr{T})d\mathscr{T} = \\ &= P_2\Big\{\int_a^b f(\lambda)d_\lambda q_2(\lambda,\mathscr{T})\Big\}. \end{aligned}$$

Lemma 1. *If $q(\lambda, t)$ is of class H on $[a, b]\times[0, \infty)$ and $x(\mathscr{T})$ is continuous on $[0, \infty)$ then $h(\lambda, t) = \int_0^t x(t-\mathscr{T})q(\lambda,\mathscr{T})d\mathscr{T}$ is of class H on $[a, b]\times[0, \infty)$.*

PROOF. Let P be any partition of $[a, b]$, $t>0$ and $\mathscr{T}>t$

$$\begin{aligned} \sum_P |h(\lambda_i,t)-h(\lambda_{i-1},t)| &\leqq \sum_P \int_0^t |x(t-\mathscr{T})|\,|q(\lambda_i,\mathscr{T})-q(\lambda_{i-1},\mathscr{T})|d\mathscr{T} \leqq \\ &\leqq M\sum_P \int_0^t |q(\lambda_i,\mathscr{T})-q(\lambda_{i-1},\mathscr{T})|d\mathscr{T} \leqq \\ &\leqq M\int_0^t \Big(\sum_P |q(\lambda_i,\mathscr{T})-q(\lambda_{i-1},\mathscr{T})|\Big)d\mathscr{T} \leqq \\ &\leqq MV\int_0^t d\mathscr{T} \leqq \\ &\leqq MVT \end{aligned}$$

Where $M=\sup|x(\mathscr{T})|$ for $\mathscr{T}$ in $[0, T]$ and V is larger than the variation of $q(\lambda, \mathscr{T})$

in λ over $[a, b]$ for $\mathcal{T}$ in $[0, T]$. Further for each λ in $[a, b]$, $h(\lambda, t)$ is continuous on $[0, \infty)$ since if $\mathcal{T} \geqq t > u > 0$ and $x(u-\mathcal{T}) = x(t-\mathcal{T}) + \varepsilon(\mathcal{T}, u)$ for all $\mathcal{T}$ in $[0, t]$ then

$$|h(\lambda, t) - h(\lambda, u)| = \left| \int_u^t x(t-\mathcal{T}) q(\lambda, \mathcal{T}) d\mathcal{T} - \int_0^u \varepsilon(\mathcal{T}, u) q(\lambda, \mathcal{T}) d\mathcal{T} \right| \leqq$$

$$\leqq M_x M_q \int_u^t d\mathcal{T} + M_q \int_0^u |\varepsilon(\mathcal{T}, u)| d\mathcal{T},$$

where $M_x = \sup |x(t)|$ for t in $[0, T]$ and $M_q \geqq |q(\lambda, t)|$ for (λ, t) in $[a, b] \times [0, T]$. Since $\varepsilon(\mathcal{T}, u) \to 0$ as $u \to t$ uniformly for u, t in $[0, T]$, we have left continuity for $h(\lambda, t)$ at each t. By similar proof we obtain right continuity. Therefore $h(\lambda, t)$ is of class H on $[a, b] \times [0, \infty)$.

Theorem 3. *If $f(\lambda)$, $f_1(\lambda)$ and $f_2(\lambda)$ are continuous on $[a, b]$, $g(\lambda)$, $g_1(\lambda)$, $g_2(\lambda)$ are of class H on $[a, b]$, c an operator, and k a complex number then*

a) $$\int_a^b f(\lambda) dcg(\lambda) = c \int_a^b f(\lambda) dg(\lambda)$$

b) $$\int_a^b kf(\lambda) dg(\lambda) = k \int_a^b f(\lambda) dg(\lambda)$$

c) $$\int_a^b f(\lambda) dg(\lambda) = \int_a^r f(\lambda) dg(\lambda) + \int_r^b f(\lambda) dg(\lambda) \quad \textit{for} \quad a < r < b.$$

d) $$\int_a^b (f_1(\lambda) + f_2(\lambda)) dg(\lambda) = \int_a^b f_1(\lambda) dg(\lambda) + \int_a^b f_2(\lambda) dg(\lambda)$$

and

e) $$\int_a^b f(\lambda) d(g_1(\lambda) + g_2(\lambda)) = \int_a^b f(\lambda) dg_1(\lambda) + \int_a^b f(\lambda) dg_2(\lambda).$$

PROOF. The proofs of parts b, c, and d are trivial. Part a follows from the fact that if $p\{q(\lambda, t)\}$ is a representation for $g(\lambda)$ then $cp\{q(\lambda, t)\}$ is a representation for $cg(\lambda)$.

To prove part e we let $g_i(\lambda) = p_i\{q_i(\lambda, t)\}$ be a representation for $g(\lambda)$ on $[a, b]$ for $i = 1, 2$ where $p_i = a_i/c$ where a_i and c are of class C (continuous on $[0, \infty)$) and $c \neq 0$. By Lemma 1 we have $a_i\{q_i(\lambda, t)\}$ is of class H on $[a, b] \times [0, \infty)$ for $i = 1, 2$. Therefore $g_1(\lambda) + g_2(\lambda)$ has a representation of $\frac{1}{c}\{a_1\{q_1(\lambda, t)\} + a_2\{q_2(\lambda, t)\}\}$ on $[a, b]$ since class H is closed under addition. The conclusion follows.

We note that the integral is a generalization of the Riemann—Stieltjes integral. That is if $g(\lambda) = \{q(\lambda, t)\}$ where $q(\lambda, t)$ is of class H on $[a, b]$ then $\int_a^b f(\lambda) dg(\lambda) = \{\int_a^b f(\lambda) d_\lambda q(\lambda, t)\}$ for all f continuous on $[a, b]$.

2. *Limits*

We use the following definition of limit which is similar to that of Mikusinski [4] for sequences of operators.

Definition 4. For $F(b)$ an operator function, $\lim\limits_{b\to\infty} F(b)=a$ if there exists an operator q such that $F(b)=q\{f(b,t)\}$ for all $b>N$ for some $N>0$, where $f(b,t)$ is continuous in t on $[0,\infty)$ for each $b>N$ and $\lim\limits_{b\to\infty} f(b,t)=h(t)$ almost uniformly on $[0,\infty)$.

This limit has the same properties as Mikusinski's sequential limits. In particular it is unique and linear.

3. *The improper Integral*

Definition 5. Operator function $g(\lambda)$ is of class H on $[a,\infty)$ if $g(\lambda)=p\{q(\lambda,t)\}$ where p is an operator and $q(\lambda,t)$ is of class H on $[a,b]\times[0,\infty)$ for all $b>a$. $p\{q(\lambda,t)\}$ is called a representation of $g(\lambda)$ on $[a,\infty)$.

Definition 6. If $f(\lambda)$ is continuous on $[a,\infty)$ and $g(\lambda)$ is of class H on $[a,\infty)$ then

$$\int_a^\infty f(\lambda)\,dg(\lambda)=\lim_{b\to\infty}\int_a^b f(\lambda)\,dg(\lambda)$$

provided the limit exists.

We note that Theorem 3 is easily extended to allow $b=\infty$.

Theorem 4. *If $f(\lambda)$ is continuous on $[a,\infty)$, $g(\lambda)$ is of class H on $[a,\infty)$ and $\int_a^\infty f(\lambda)\,dg(\lambda)$ exists then $g(\lambda)$ has a representation $p\{q(\lambda,t)\}$ on $[a,\infty)$ such that $\lim\limits_{b\to\infty}\int_a^b f(\lambda)\,dq(\lambda,t)$ exists almost uniformly and $\int_a^b f(\lambda)\,dg(\lambda)=p\Big\{\int_a^\infty f(\lambda)\,d_\lambda q(\lambda,t)\Big\}$.*

Proof. Let $c\{q(\lambda,t)\}$ be a representation of $g(\lambda)$ on $[a,\infty)$. Then $\int_a^\infty f(\lambda)\,dg(\lambda)=$ $=\lim\limits_{b\to\infty} c\Big\{\int_a^b f(\lambda)\,d_\lambda q(\lambda,t)\Big\}$ implies the existence of an operator p and a function $F(b,t)$ continuous in t on $[0,\infty)$ for each $b>N$ for some $N>0$ such that $\lim\limits_{b\to\infty} F(b,t)=$ $=h(t)$ almost uniformly on $[0,\infty)$ and $c\Big\{\int_a^b f(\lambda)\,d_\lambda q(\lambda,t)\Big\}=p\{F(b,t)\}$. Let $c=c_1/e$ and $p=p_1/e$ where c_1, p_1 and e are of class C (continuous on $[0,\infty)$) and $e\neq 0$. Then $c_1\Big\{\int_a^b f(\lambda)\,d_\lambda q(\lambda,t)\Big\}=p_1\{F(b,t)\}$. Clearly $\lim\limits_{b\to\infty} p_1\{F(b,t)\}=p_1\{h(t)\}$ almost uniformly on $[0,\infty)$. Therefore

$$\int_a^\infty f(\lambda)dg(\lambda) = p_1/e\{h(t)\} =$$
$$= p_1/e\Big\{\lim_{b\to\infty} F(b,t)\Big\} =$$
$$= 1/e\Big\{\lim_{b\to\infty} p_1\{F(b,t)\}\Big\} =$$
$$= 1/e\Big\{\lim_{b\to\infty} c_1\Big\{\int_a^b f(\lambda)d_\lambda q(\lambda,t)\Big\}\Big\} =$$
$$= 1/e\Big\{\lim_{b\to\infty}\int_a^b f(\lambda)d_\lambda c_1\{q(\lambda,t)\}\Big\}.$$

Since $q(\lambda, t)$ is of class H on $[a,\infty)\times[0,\infty)$, $c_1\{q(\lambda,t)\}$ is of class H on $[a,\infty)\times\times[0,\infty)$ by Lemma 1. Therefore $1/e\{c_1\{q(\lambda,t)\}\}$ is the desired representation of $g(\lambda)$ on $[a,\infty)$.

4. *The transform*

Definition 7. If $f(\lambda)=f_1(\lambda)+if_2(\lambda)$ and $g(\lambda)$ is an operator function we take
$$\int_a^b f(\lambda)dg(\lambda) = \int_a^b f_1(\lambda)dg(\lambda)+i\int_a^b f_2(\lambda)dg(\lambda)$$
provided $f_1(\lambda)$ and $f_2(\lambda)$ are integrable with respect to $g(\lambda)$. We allow $b=\infty$ in this definition.

Definition 8. If $g(\lambda)$ is of class H on $[0,\infty)$ and r is a complex number such that $\int_0^\infty e^{-\lambda r}dg(\lambda)$ exists then we call this integral the Laplace—Stieltjes transform of $g(\lambda)$ and denote it as $L(g(\lambda))$.

That the transform is well defined linear, and a generalization of the usual Laplace—Stieltjes transform follows from the properties of the improper integral.

Using the definition of MIKUSINSKI for the continuous derivative $g'(\lambda)$ of operator function $g(\lambda)$ for interval $[0,\infty)$ we have the following theorem.

Theorem 5. *If $g(\lambda)$ has a representation $p\{q(\lambda,t)\}$ on $[0,\infty)$ such that $q_\lambda(\lambda,t)$ is continuous on $[0,\infty)\times[0,\infty)$, $\int_0^\infty e^{-r\lambda}d_\lambda g_\lambda(\lambda,t)$ converges almost uniformly for $0\leqq t<\infty$ and $\lim_{\lambda\to\infty} e^{-\lambda r}q_\lambda(\lambda,t)=0$ almost uniformly for t in $[0,\infty)$ then*
$$L(g'(\lambda)) = -g'(0)+rL(g(\lambda)).$$

PROOF. $\int_0^\infty e^{-\lambda r}d_\lambda q_\lambda(\lambda,t) =$
$$= \lim_{b\to\infty}\Big[e^{-\lambda r}q_\lambda(\lambda,t)|_0^b + r\int_0^b e^{-\lambda r}q_\lambda(\lambda,t)d_\lambda\Big]$$

$$= -q_\lambda(0, t) + r\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$$

$\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)$ exist almost uniformly on $[0, \infty)$ since $\int_0^\infty e^{-\lambda r} d_\lambda q_\lambda(\lambda, t)$ and $\lim_{b\to\infty} e^{-\lambda b} q_\lambda(b, t)$ exist almost uniformly. Hence

$$L(g'(\lambda)) = p\left\{-q_\lambda(0, t) + r\int_0^\infty e^{-\lambda r} d_\lambda q(\lambda, t)\right\} = -g'(0) + rL(g(\lambda)).$$

5. *Transforms of specific function*

Consider the heaviside function

$$H_\lambda(t) = \begin{Bmatrix} 0, & 0 \leqq t \leqq \lambda \\ 1, & \lambda < t \end{Bmatrix}.$$

Theorem 6. *If* $H(\lambda) = \{H_\lambda(t)\}$ *then* $L(H(\lambda)) = \dfrac{-1}{s+r}$.

PROOF. $H_\lambda = s\{h_1(\lambda, t)\}$ where $h_1(\lambda, t) = \begin{Bmatrix} 0, & 0 \leqq t \leqq \lambda \\ t-\lambda & \lambda < t \end{Bmatrix}$.
Clearly $h_1(\lambda, t)$ is of class H on $[0, b]\times[0, \infty)$ for all $b > 0$, therefore

$$\int_0^b e^{-\lambda r} d_\lambda H(\lambda) = s\left\{\int_0^b e^{-\lambda r} d_\lambda h_1(\lambda, t)\right\} = s\{g_b(r, t)\}$$

where

$$g_b(r, t) = \begin{Bmatrix} \dfrac{e^{-tr}-1}{r}, & t \leqq b \\ \dfrac{e^{-br}-1}{r}, & b < t \end{Bmatrix}$$

$g_b(r, t)$ is continuous for t in $[0, \infty)$ if $r \neq 0$ $b > 0$. Further for $T > 0$ if $b > T$ and t in $[0, T]$ then

$$g_b(r, t) = \frac{e^{-tr}-1}{r}$$

and $g_b(r, t)$ has a uniform limit as $b \to \infty$. Therefore

$$\int_0^\infty e^{-\lambda r} d_\lambda H(\lambda) = s\left\{\frac{e^{-tr}-1}{r}\right\} = -\{e^{-tr}\} = \frac{-1}{s+r}$$

if $r \neq 0$. If $r = 0$ for $b > 0$ we have

$$\int_0^b e^{-\lambda r} dH(\lambda) = \int_0^b dH(\lambda) = s\{f_b(t)\}$$

where $f_b(t) = \begin{Bmatrix} -t, & t \leqq b \\ -b, & b < t \end{Bmatrix}$ is continous for $b > 0$, and $t > 0$ and $\lim_{b \to \infty} f_b(t) = -t$ almost uniformly on $[0, \infty)$. Therefore

$$\int_0^b dH(\lambda) = s\{-t\} = -\{1\} = \frac{-1}{s} = \frac{-1}{s+r}$$

since $r = 0$.

Next we consider the translation operator $h(\lambda) = s\{H_\lambda(t)\}$.

Corollary 8a. $L(h(\lambda)) = \dfrac{-s}{s+r}$ for all r.

Proof. Follows from the linearity of the transform and Theorem 6.

6. *Application*

We define the inverse transform by $L^{-1}(h(r)) = g(\lambda)$ if $L(g(\lambda)) = h(r)$, and note that the transform may be used to solve operational differential equations.

As an example consider $x''(\lambda) = -sx'(\lambda)$. Proceeding formally and using Theorem 5 we obtain $L(x(\lambda)) = \dfrac{A}{r} + \dfrac{B}{r+s}$ where A and B depend on $x'(0)$ and $x''(0)$. Taking the inverse transform we obtain $x(\lambda) = A\lambda - BH(\lambda)$.

II.

Introduction

In I. we defined a Laplace—Stieltjes transform for operator valued functions of a complex variable. We will extend this work further by considering convergence and operational properties for the transform.

1. *Convergence theory and order properties*

Notation

If we write $L(g(\lambda)) = p\left\{\int_0^\infty e^{-\lambda r} dq(\lambda, t)\right\}$ exists from some r then $g(\lambda) = p\{q(\lambda, t)\}$ for $\lambda \in [0, \infty)$, where p is an operator, $q(\lambda, t) \in H$ on $[0, \infty) \times [0, \infty)$ (i.e. on $[0, b] \times \times [0, \infty)$ for all $b > 0$), and $\int_0^\infty e^{-\lambda r} dq(\lambda, t)$ exists almost uniformly with respect to t on $[0, \infty)$.

We note that the existence of the transform implies the existence of operator p and function $q(\lambda, t)$ with the above properties by Theorem 4 and Definition 8 in I.

We have the following generalization of the usual theorem concerning the region of convergence of the transform.

Theorem 1. *If* $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r}\, dq(\lambda, t)\right\}$ *exists for* $r_0 = a_0 + ib_0$ *then* $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r}\, dq(\lambda, t)\right\}$ *exsits for all* $r = a + ib$ *with* $a > a_0$.

Proof. Let $B(u, t) = \int\limits_0^u e^{-\lambda r_0}\, dq(\lambda, t)$ and $T > 0$. Since $\lim\limits_{u \to \infty} B(u, t) = h(t)$ uniformly on $[0, T]$ and $q(\lambda, t)$ is of uniformly bounded variation in λ on any interval of the form $[0, N]$ for t in $[0, T]$ and continuous in t on $[0, T]$ for each $\lambda \geqq 0$, there exists a $J > 0$ such that $|B(b, t)| < J$ for all $(b, t) \in [0, \infty) \times [0, T]$. Further,

$$\int\limits_0^b e^{-\lambda r}\, d_\lambda q(\lambda, t) = \int\limits_0^b e^{-\lambda(r - r_0)}\, d_\lambda B(\lambda, t) =$$

$$= e^{-b(r-r_0)} B(b, t) + (r - r_0) \int\limits_0^b e^{-\lambda(r-r_0)} B(\lambda, t)\, d\lambda.$$

Since $a > a_0$ and $|B(b, t)| < J$ for $(b, t) \in [0, \infty) \times [0, T]$, we have $e^{-b(r-r_0)} B(b, t) \to 0$ uniformly on $[0, T]$ as $b \to \infty$.

Also for $t \in [0, T]$,

$$\left| \int\limits_b^\infty e^{-\lambda(r-r_0)} B(\lambda, t)\, d\lambda \right| \leqq J \int\limits_b^\infty e^{-\lambda(r-r_0)}| d\lambda \leqq \frac{J}{a - a_0} e^{-b(a-a_0)},$$

which approaches 0 as $b \to \infty$. Therefore $\int\limits_0^\infty e^{-\lambda r}\, d_\lambda q(\lambda, t)$ converges almost uniformly on $[0, \infty)$ for $a > a_0$.

Theorem 2. *If* $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r}\, dq(\lambda, t)\right\}$ *exists for* $r = a + ib$ *then*

(1) $a > 0$ *implies* $\lim\limits_{\lambda \to \infty} q(\lambda, t) e^{-a\lambda} = 0$ *almost uniformly on* $[0, \infty)$

and

(2) $a < 0$ *implies* $\lim\limits_{\lambda \to \infty} q(\lambda, t) = q(\infty, t)$ *exists and* $\lim\limits_{\lambda \to \infty} \left(q(\infty, t) - q(\lambda, t)\right) e^{-a\lambda} = 0$ *almost uniformly on* $[0, \infty)$.

Proof. If $a > 0$ let $B(u, t) = \int\limits_0^u e^{-rw}\, d_w q(w, t)$ then

$$q(\lambda, t) - q(0, t) = \int\limits_0^\lambda d_\lambda q(\lambda, t) = \int\limits_0^\lambda e^{ru}\, d_u B(u, t).$$

Integrating by parts, multiplying by $e^{-r\lambda}$ rearranging and taking the limit of both sides of the equation gives us

$$\lim_{\lambda\to\infty}\big(q(\lambda,t)-q(0,t)\big)e^{-r\lambda} = B(\infty,t)-\lim_{\lambda\to\infty} re^{-\lambda r}\int_0^\lambda e^{ru}B(u,t)\,du =$$

$$= \lim_{\lambda\to\infty} re^{-r\lambda}\int_0^\lambda e^{ru}[B(\infty,t)-B(u,t)]\,du.$$

Since $\lim_{\lambda\to\infty} B(\lambda,t)=B(\infty,t)$ exists almost uniformly on $[0,\infty)$, for $\varepsilon>0$, $T>0$ there exists a J such that if $\lambda\geqq J$ then

$$|B(\lambda,t)-B(\infty,t)|<\varepsilon \quad \text{for all} \quad t\in[0,T]. \quad \text{For} \quad \lambda>J,$$

$$\int_0^\lambda e^{ru}\big(B(\infty,t)-B(u,t)\big)du = \int_0^J e^{ru}\big(B(\infty,t)-B(u,t)\big)du+\int_J^\lambda e^{ru}\big(B(\infty,t)-B(u,t)\big)du.$$

However,

$$\left|re^{-r\lambda}\int_0^J e^{ru}\big(B(\infty,t)-B(u,t)\big)du\right| \leqq |r|e^{-\lambda a}e^{aJ}MJ,$$

where M is an upper bound of $|B(\infty,t)-B(u,t)|$ for $0\leqq u\leqq J$, $0\leqq t\leqq T$. Also,

$$\left|\int_J^\lambda e^{ru}\big(B(\infty,t)-B(u,t)\big)du\right| \leqq \int_J^\lambda e^{au}\varepsilon\,du \leqq \frac{e^{a\lambda}}{a}\varepsilon.$$

Choose $K>J$ such that if $\lambda>K$ then $|r|e^{-a\lambda}e^{aJ}MJ<\varepsilon$. Then for $\lambda>K$, $t\in[0,T]$ we have

$$\left|re^{-\lambda r}\int_0^\infty e^{ru}\big(B(\infty,t)-B(u,t)\big)du\right| \leqq \varepsilon+\frac{|r|}{a}\varepsilon.$$

Therefore $\lim_{\lambda\to\infty}\big(q(\lambda,t)-q(0,t)\big)e^{-\lambda r}=0$ and hence $\lim_{\lambda\to\infty} q(\lambda,t)e^{-\lambda r}=0$.

If $a<0$ by Theorem 1 we have $L\big(g(\lambda)\big)$ exists at $r=0$ or $\int_0^\infty d_\lambda q(\lambda,t) = q(\infty,t)+\big(-q(0,t)\big)$ almost uniformly for $t\in[0,\infty)$. Therefore $q(\infty,t)$ exists. Further since $a<0$.

$$q(\infty,t)-q(\lambda,t)=\int_\lambda^\infty d_u q(u,t)=\int_\lambda^\infty e^{ru}d_u B(u,t) = e^{r\lambda}B(\lambda,t)-r\int_\lambda^\infty e^{ru}B(u,t)\,du.$$

Therefore

$$\lim_{\lambda\to\infty}\big(q(\infty,t)-q(\lambda,t)\big)e^{-r\lambda} = B(\infty,t)-\lim_{\lambda\to\infty} re^{-\lambda r}\int_\lambda^\infty e^{ru}B(u,t)\,du =$$

$$= \lim_{\lambda\to\infty} re^{-\lambda r}\int_\lambda^\infty e^{ru}\big(B(\infty,t)-B(u,t)\big)du.$$

For $\varepsilon>0$, $T>0$ we choose J as before such that if $\lambda\geqq J$,

$|B(\infty, t)-B(\lambda, t)|<\varepsilon$ for $0 \leqq t \leqq T$. Then for $\lambda \geqq J$,

$$\left|re^{-r\lambda}\int_{\lambda}^{\infty} e^{ru}\big(B(\infty, t)-B(u, t)\big)du\right| \leqq |r|e^{-a\lambda}\varepsilon\int_{\lambda}^{\infty} e^{au}du \leqq \frac{|r|}{|a|}\varepsilon.$$

Therefore $\lim\limits_{\lambda\to\infty}|(q(\infty, t)-q(0, t))e^{-\lambda r}| = 0$ and hence $\lim\limits_{\lambda\to\infty}\big(q(\infty, t)-q(0, t)\big)e^{-\lambda a} = 0$ almost uniformly on $[0, \infty)$.

Definition 1. If a is any real number, $g(\lambda)$ an operator valued function is said to be of order $a^{a\lambda}$ if there is an operator p such that $g(\lambda)=p\{q(\lambda, t)\}$ where $q(\lambda, t)\in H$ on $[0, \infty)\times[0, \infty)$ and $|q(\lambda, t)|\leqq m(t)e^{a\lambda}$ where $m(t)$ is bounded on $[0, T]$ for all $T>0$.

Theorem 3. *If $g(\lambda)$ is an operator function of order $e^{a\lambda}$ then $L\big(g(\lambda)\big)$ exists for all $r = c+di$, with $c>a$.*

PROOF. Let $g(\lambda)=p\{q(\lambda, t)\}$ where $q(\lambda, t)$ has the properties of $q(\lambda, t)$ in definition 1. For $T>0$,

$$\int_0^R e^{-\lambda r}d_\lambda q(\lambda, t) = q(R, t)e^{-Rr}-q(0, t)+r\int_0^R e^{-\lambda r}q(\lambda, t)\,d\lambda.$$

But $|q(R, t)e^{-rR}| \leqq m(t)e^{aR}|e^{-rR} \leqq M_T e^{-(c-a)R}$ where M_T is an upper bound for $m(t)$ on $[0, T]$. Therefore $\lim\limits_{R\to\infty} q(R, t)e^{-rR} = 0$ uniformly on $[0, T)$. Further

$$\int_0^R \left|e^{-\lambda r}q(\lambda, t)\right|d\lambda \leqq \int_0^R e^{-\lambda(c-a)}m(t)\,d\lambda \leqq M_T\frac{e^{-R(c-a)}-1}{-(c-a)}.$$

Therefore $\int_0^\infty e^{-\lambda r}q(\lambda, t)\,d\lambda$ and hence $\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda, t)$ converge almost uniformly for $t\in[0, \infty)$.

Theorem 4. *If $\lim\limits_{\lambda\to\infty} g(\lambda)=g(\infty)$ exists and $g(\lambda)-g(\infty)$ is of order $e^{a\lambda}$ then $L\big(g(\lambda)\big)$ exists for all $r = c+id$ with $c>a$.*

PROOF. $L\big(g(\lambda)-g(\infty)\big)$ exists by Theorem 3.

$$L\big(g(\lambda)-g(\infty)\big) = \int_0^\infty e^{-\lambda r}d\big(g(\lambda)-g(\infty)\big) = \int_0^\infty e^{-\lambda r}dg(\lambda) = L\big(g(\lambda)\big).$$

Theorem 5. *If $L\big(g(\lambda)\big) = p\Big\{\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda, t)\Big\}$ exists for $r = a+bi$ then*

1) *$a>0$ implies $g(\lambda)$ is of order $e^{a\lambda}$ and*

2) *$a<0$ implies $g(\lambda)-g(\infty)$ is order $e^{a\lambda}$.*

PROOF. By Theorem 2 for $T>0$ there is a J such that if $\lambda>J$ then for $t\in[0, T]$, $|q(\lambda, t)e^{-a\lambda}|<1$. Further $|q(\lambda, t)|<M$ for $\lambda\in[0, J]$, $t\in[0, T]$ since $q(\lambda, t)$ is of uniformly bounded variation in λ on $[0, J]$ for $t\in[0, T]$. Let $m(T)=\text{lub}\{M\,\big|\,|q(\lambda, t)|\leqq$

$\leqq Me^{a\lambda}$ for $\lambda \geqq 0$, $t \in [0, T]\}$ then $|q(\lambda, t)| \leqq m(t)e^{a\lambda}$ and $m(t)$ is bounded on every bounded interval $[0, T]$.

The proof of part 2 is similar and will be omitted.

Theorem 6. *If* $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r}\, dq(\lambda, t)\right\}$ *exists for some* $r = a+bi$ *and* $h(r, t) = \int\limits_0^\infty e^{-\lambda r}\, dq(\lambda, t)$ *then*

1) *If* $a>0$ $h(r, t) = r\int\limits_0^\infty e^{-\lambda r} q(\lambda, t)\, d\lambda - q(0, t)$ *and* $\int\limits_0^\infty e^{-\lambda r} q(\lambda, t)\, d\lambda$ *converges absolutely and almost uniformly for* $t \in [0, \infty)$ *for all* $r = c+di$ *with* $c>a$, *and*

2) *if* $a<0$ $h(r, t) = q(\infty, t) - q(0, t) + r\int\limits_0^\infty e^{-\lambda r}\big(q(\lambda, t) - q(\infty, t)\big)\, d\lambda$

and $\int\limits_0^\infty e^{-\lambda r}\big(q(\lambda, t) - q(\infty, t)\big)\, d\lambda$ *converges absolutely and almost uniformly for* $t \in [0, \infty)$ *for all* $r = c+di$ *with* $c>a$.

Proof. If $a>0$, by Theorem 5, $g(\lambda)$ is of order $e^{a\lambda}$, and in the proof of Theorem 3 we have already proven that $h(r, t) = r\int\limits_0^\infty e^{-\lambda r} q(\lambda, t)\, d\lambda - q(0, t)$ and that the integral converges absolutely and almost uniformly.

If $a<0$, after using Theorem 5, the proof is similar to that of Theorem 3 and will be omitted.

Definition 2. If $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r}\, d_\lambda q(\lambda, t)\right\}$ exists for each $r \in A$ we say $L(g(\lambda))$ exists uniformly on A if $\int\limits_0^\infty e^{-\lambda r}\, d_\lambda q(\lambda, t)$ converges uniformly for $r \in A$ and almost uniformly for $t \in [0, \infty)$.

Theorem 7. *If* $L(g(\lambda)) = p\left\{\int\limits_0^\infty e^{-\lambda r_0}\, d_\lambda q(\lambda, t)\right\}$ *exists for* $r_0 = a_0 + b_0 i$, $H>0$ *and* $K>1$ *then* $L(g(\lambda))$ *exists uniformly on* $A = \{r | r = a+bi$ *such that* $|r - r_0| \leqq K(a-a_0)e^{H(a-a_0)}$ *and* $a>a_0\}$.

Proof. If $r = a+bi \in A$ and $a>a_0$ the $L(g(\lambda))$ exists by Theorem 1 and if $a=a_0$ then $r=r_0$ and $L(g(\lambda))$ exists. For $\varepsilon>0$, $T>0$ we must show the existence of R_0 such that if $R>R_0$

$$\left|\int\limits_R^\infty e^{-\lambda r}\, d_\lambda q(\lambda, t)\right| < \varepsilon \quad \text{for all} \quad r \in A \quad \text{and} \quad t \in [0, T].$$

Let $B(\lambda, t) = \int\limits_0^\infty e^{-\lambda r_0}\, d_\lambda q(\lambda, t)$, since $B(\infty, t)$ exists uniformly on $[0, T]$ pick $R_0 > H$ such that

$$|B(w, t) - B(u, t)| < \varepsilon/K \quad \text{if} \quad w, u > R_0 \quad \text{for all } t \text{ in} \quad [0, T].$$

For $R>R_0$, $r=a+bi\in A$ with $a>a_0$ we have

$$\Big|\int_R^\infty e^{-\lambda r}d_\lambda q(\lambda,t)\Big| = \Big|\int_R^\infty e^{-(r-r_0)}d_\lambda\big(B(\lambda,t)-B(R,t)\big)\leqq$$

$$\leqq |r-r_0|\int_R^\infty e^{-\lambda(r-r_0)}||B(\lambda,t)-B(R,t)|d\lambda\leqq$$

$$\leqq |r-r_0|\varepsilon/K\int_R^\infty e^{-(a-a_0)\lambda}d\lambda\leqq$$

$$\leqq K(a-a_0)e^{H(a-a_0)}\varepsilon/K\frac{e^{-(a-a_0)R}}{a-a_0}\leqq$$

$$\leqq \varepsilon,$$

since $H<R$ and $a>a_0$. If $a=a_0$ then

$$\Big|\int_R^\infty e^{-r\lambda}d_\lambda q(\lambda,t)\Big| = |B(\infty,t)-B(R,t)|\leqq \varepsilon/K<\varepsilon.$$

The above inequalities are clearly independent of $t\in[0,T]$.

Definition 3. $L(g(\lambda))=p\Big\{\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda,t)\Big\}$ exists absolutely for some r if $\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda,t)$ converges absolutely and almost uniformly on $[0,\infty)$.

Theorem 8. *If* $L(g(\lambda))=p\Big\{\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda,t)\Big\}$ *exists absolutely for* $r=a+bi$ *then it exists absolutely and uniformly for all* $r=c+di$ *with* $c\geqq a$.

Proof. Let $Vq(\lambda,t)$ be the variation of $q(u,t)$ for $0\leqq u\leqq\lambda$ for each $t\geqq 0$, then for $T>0$, $\varepsilon>0$ there is an R such that $\int_R^\infty e^{-\lambda a}d_\lambda Vq(\lambda,t)<\varepsilon$ for $t\in[0,T]$. Therefore if $c\geqq a$, $r=c+di$ then $\int_R^\infty |e^{-\lambda r}|d_\lambda Vq(\lambda,t)<\varepsilon$ for all $t\in[0,T]$.

2. *Operational properties*

Theorem 9. *If* $L(g(\lambda))=p\Big\{\int_0^\infty e^{-\lambda r}dq(\lambda,t)\Big\}=h(r)$ *exists for some* r, $c>0$ *then* $L(g(\lambda/c))=h(cr)$.

Proof. $g(\lambda/c)=p\{q(\lambda/c,t)\}$ and $q(\lambda/c,t)\in H$ on $[0,\infty)\times[0,\infty)$. Using $uc=\lambda$ we obtain

$$p\Big\{\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda/c,t)=p\Big\{\int_0^\infty e^{-urc}d_u q(u,t)\Big\}=h(cr).$$

Theorem 10. *If $L(g(\lambda))=h(r)$ exists for $r=a+bi$ then for $r=c+di$, $c>a$, and k any positive integer*

$$h^{(k)}(r)=\int_0^\infty (-\lambda)^k e^{-\lambda r}\,dg(\lambda).$$

PROOF. Let $L(g(\lambda))=p\left\{\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda,t)\right\}$ exist for $r=a+bi$ and $h_1(r,t)=$ $=\int_0^\infty e^{-\lambda r}d_\lambda q(\lambda,t)$. By Theorem 5 in WIDDER [5] we have $\frac{\partial^k}{\partial r^k}(h_1(r,t))=$

$$=\int_0^\infty -(\lambda)^k e^{-\lambda r}d_\lambda(q(\lambda,t)) \text{ for each } t\geqq 0.$$

We know $\frac{\partial^k h_1}{\partial r^k}$ is continuous as a function of r since the $\frac{\partial^{k+1}h_1}{\partial r^{k+1}}$ exists. We need only show the almost uniform convergence of this integral to get that it is continuous as a function of t.

We consider the convergence problem for $k=1$.

i) If $c>a>0$ we have

$$\int_0^\infty (-\lambda)e^{-\lambda r}d_\lambda q(\lambda,t)=-\lambda e^{-\lambda r}q(\lambda,t)|_0^\infty-\int_0^\infty q(\lambda,t)d_\lambda(-\lambda e^{-\lambda r}).$$

But $\lambda e^{-\lambda r}q(\lambda,t)\to 0$ almost uniformly for $t\in[0,\infty)$ as $\lambda\to\infty$ and

$$-\int_0^\infty q(\lambda,t)d_\lambda(-\lambda e^{-\lambda r})=\int_0^\infty q(\lambda,t)e^{-\lambda r}d\lambda-r\int_0^\infty \lambda q(\lambda,t)e^{-\lambda r}d\lambda.$$

However $\int_0^\infty q(\lambda,t)e^{-\lambda r}d\lambda$ exists almost uniformly on $[0,\infty)$ by Theorem 6. By Theorem 5 $\int_R^\infty |\lambda e^{-\lambda r}q(\lambda,t)|\,d\lambda\leqq\int_R^\infty \lambda e^{-\lambda c}m(t)e^{\lambda a}d\lambda$ where $m(t)$ is bounded on $[0,T]$ for any $T>0$. The integral on the right converges to 0 as $R\to\infty$ uniformly on $[0,T]$. Therefore $\int_0^\infty \lambda e^{-\lambda r}q(\lambda,t)d\lambda$ converges almost uniformly on $[0,\infty)$.

ii) If $c>a\geqq 0$ then $L(g(\lambda))$ exists for $r=a_1$ for some $0<a_1<c$ and the problem reduces to case i).

iii) If $a<0$ and $a<c$ then

$$\int_0^\infty (-\lambda)e^{-\lambda r}d_\lambda q(\lambda,t)=\int_0^\infty (-\lambda)e^{-\lambda r}d_\lambda(q(\lambda,t)-q(\infty,t))$$

and the proof is as in case i) with $q(\lambda,t)$ replaced by $q(\lambda,t)-q(\infty,t)$.

Using induction and similar arguments to those in the case of $k=1$ it is easily shown that convergence is almost uniform on $[0,\infty)$ for all positive integers k.

Multiplying $\frac{\partial^k h_1}{\partial r^k}$ by p completes the proof.

Definition 4. If $g(\lambda)\in H$ on $[a,b]$, $f(\lambda)\in C[a,b]$ then $\int\limits_a^b g(\lambda)df(\lambda) = g(b)f(b)-$ $-g(a)f(a)-\int\limits_a^b f(\lambda)dg(\lambda)$. If $g(\infty)$, $f(\infty)$ and $\int\limits_a^\infty f(\lambda)dg(\lambda)$ extists we allow $b=\infty$ in the above definition (which of course changes $[a,b]$ to $[a,\infty)$).

Theorem 11. *If $L(g(\lambda))=h(r)$ exists for $r_0=a+bi$ then*

$$1)\qquad a<0 \quad \textit{implies} \quad L\Big(\int\limits_0^\lambda q(u)\,du\Big)=\frac{h(r_0)+g(0)}{r_0} \quad \textit{for} \quad r=r_0$$

and

$$2)\quad a<0 \quad \textit{implies} \quad L\Big(\int\limits_0^\lambda (g(\lambda)-g(\infty))\,d\lambda\Big)=\frac{h(r_0)+g(0)-g(\infty)}{r_0} \quad \textit{for} \quad r=r_0.$$

Proof. Let $h(r_0)=p\{h_1(r_0,t)\}$ where $g(\lambda)=p\{q(\lambda,t)\}$ and

$$h_1(r_0,t)=\int\limits_0^\infty e^{-\lambda r_0}d_\lambda q(\lambda,t)=$$

$$=r_0\int\limits_0^\infty e^{-\lambda r_0}q(\lambda,t)\,d\lambda-q(0,t)=$$

$$=r_0\int\limits_0^\infty e^{-\lambda r_0}d_\lambda\Big[\int\limits_0^\lambda q(\lambda,t)\,d\lambda\Big]-q(0,t),$$

and

$$\frac{h_1(r_0,t)+q(0,t)}{r_0}=\int\limits_0^\infty e^{-\lambda r_0}d_\lambda\Big(\int\limits_0^\lambda q(\lambda,t)\Big)\,d\lambda\Big)$$

almost uniformly on $[0,\infty)$. Then

$$p\frac{\{h_1(r_0,t)+q(0,t)\}}{r_0}=\int\limits_0^\infty e^{-\lambda r_0}d\lambda\Big(p\Big\{\int\limits_0^\lambda q(\lambda,t)\,d\lambda\Big\}\Big)$$

$$\frac{h(r_0)+g(0)}{r_0}=\int\limits_0^\infty e^{-\lambda r_0}d_\lambda\Big(p\Big\{\lambda q(\lambda,t)-\int\limits_0^\lambda \lambda d_\lambda q(\lambda,t)\Big\}\Big)=$$

$$=\int\limits_0^\infty e^{-\lambda r_0}d_\lambda\Big(\lambda g(\lambda)-\int\limits_0^\lambda \lambda dg(\lambda)\Big)=\int\limits_0^\infty e^{-\lambda r_0}d\Big(\int\limits_0^\lambda g(\lambda)\,d\lambda\Big).$$

The proof of 2) is similar and will be omitted.

Theorem 12. *If $L(g(\lambda))=h(r)$ for some $r_0=a+ib$, then*

1) $a\geqq 0$ *implies* $L\big(\lambda(g(\lambda)-g(\infty))\big)=\dfrac{h(r)+g(0)}{r}-h'(r)$ *for* $r=c+di$ *with* $c>a$ *and*

2) $a<0$ *implies* $L(\lambda(g(\lambda)-g(\infty)))=\dfrac{h(r)+g(0)-g(\infty)}{r}-h'(r)$ *for* $r=c+di$ *with* $a<c<0$.

Proof. If $c>a>0$ let $h(r)=p\left\{\int\limits_0^\infty e^{-\lambda r}dq(\lambda,t)\right\}$, then $\lambda g(\lambda)=p\{\lambda q(\lambda,t)\}$ and

$$p\left\{\int\limits_0^\infty e^{-\lambda r}d_\lambda(\lambda q(\lambda,t))\right\}=p\left\{\int\limits_0^\infty e^{-\lambda r}q(\lambda,t)\,d\lambda+\int\limits_0^\infty e^{-\lambda r}\lambda d_\lambda q(\lambda,t)\right\}=$$

$$=\frac{h(r)+g(0)}{r}-h'(r)$$

by Theorems 10 and 11.

If $a<0$ the proof is similar to the above.

Theorem 13. *If* $L(g(\lambda))=h(r)$ *for* $r=a_0+ib_0$ *then*

1) $a>0$ *implies* $L(e^{a\lambda}g(\lambda))=\dfrac{rh(r-a)+ag(0)}{r-a}$ *for* $\operatorname{Re}(r-a)>a_0$

and

2) $a<0$ *implies* $L(e^{a\lambda}(g(\lambda)-g(\infty)))=\dfrac{rh(r-a)+a[g(0)-g(\infty)]}{r-a}$ *for* $a_0<\operatorname{Re}(r-a)<0$.

Proof. Let $h(r)=p\left\{\int\limits_0^\infty e^{-\lambda r}dq(\lambda,t)\right\}$ and $a_0>0$ then

$$\int\limits_0^\infty e^{-\lambda r}d_\lambda(e^{a\lambda}q(\lambda,t))=\int\limits_0^\infty e^{-\lambda(r-a)}d_\lambda q(\lambda,t)+a\int\limits_0^\infty e^{-(r-a)}d_\lambda\left(\int\limits_0^\lambda q(\lambda,t)\,d\lambda\right).$$

Multiplying by p and using Theorem 11 completes the proof of part 1. The proof of 2) is similar and will be omitted.

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