

Basic submodules of an R -module

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KULIKOV introduced the notion of basic subgroup of an abelian p -group which has proved to be one of the most important notions in the theory of p -groups of arbitrary power.

Basic submodules can be defined in any module over the ring of p -adic integers, or, more generally, over any discrete valuation ring. FUCHS (Notes on abelian group II. *Acta Math. Acad. Sci. Hung.* **11** (1960), 117—125) has given a generalization of basic subgroups to any group so that it coincides with the old concept whenever the group is primary. In the general case of groups, to every prime p , one can define p -basic subgroups, where in the definition the prime p plays a distinguished role. In this paper, it is our aim to generalize this concept of basic subgroups in any group to basic submodule in a module over a Dedekind ring.

A Dedekind ring R is an integral domain in which every ideal ($\neq 0$, $\neq R$) is uniquely a product of prime ideals. Moreover, every prime ideal of a Dedekind ring is maximal and every ideal is finitely generated. During our investigation here, we will consider R to be a Dedekind ring and we shall exclusively consider modules over the fixed Dedekind ring R which we shall call for the sake of brevity R -modules.

Definition 1: The nonzero elements $a_1, a_2, \dots, a_n$ of an R -module M will be called *linearly independent* or simply *independent*, if

$$(1) \quad \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0 \quad (\lambda_i \in R)$$

implies

$$(2) \quad \lambda_1 a_1 = 0, \dots, \lambda_n a_n = 0$$

If $O(a)$ denotes the order of $a \in M$, i.e. the ideal of R consisting of all $\lambda \in R$ with $\lambda a = 0$, then the conclusion (2) can also be written in the form

$$\lambda_1 \in O(a_1), \dots, \lambda_n \in O(a_n).$$

If every element is of order 0, then naturally,

$$\lambda_1 = 0, \dots, \lambda_n = 0.$$

Remark 1: An infinite set of elements is independent if each of its finite subsets is independent, therefore, the independence of the set $\{a_\lambda\}_{\lambda \in A}$ is equivalent to the condition that the submodule $\langle \dots, a_\lambda, \dots \rangle$ generated by all the a_λ is the direct sum of the cyclic submodules Ra_λ :

$$\langle \dots, a_\lambda, \dots \rangle = \dots \oplus Ra_\lambda \oplus \dots$$

2. If each of the a_λ ($\lambda \in A$) is of order 0, then the independence of the set $\{a_\lambda\}_{\lambda \in A}$ is equivalent to the condition that the submodule $\langle \dots, a_\lambda, \dots \rangle$ is a free R -module with a_λ as free generators.

3. Independence is a property of finite character. Therefore, in every R -module M there exist maximal independent sets. Moreover, every independent set of M can be extended to a maximal independent set of M .

Definition 2: Let M be an R -module and P a fixed prime ideal of R . We shall call a subset $\{a_\lambda\}_{\lambda \in A}$ of M not containing zero, P -independent, if for any finite subset $a_1, a_2, \dots, a_k$, a relation

$$n_1 a_1 + \dots + n_k a_k \in P^r M \quad (n_i a_i \neq 0, n_i \in R)$$

implies $\langle n_i \rangle \subseteq P^r$, where r is any positive integer, $i=1, 2, \dots, k$.

Lemma 1. *A P -independent set is independent.*

PROOF. Suppose $\{a_\lambda\}_{\lambda \in A}$ is P -independent and $n_1 a_1 + \dots + n_k a_k = 0$, where $n_i a_i \neq 0$ ($n_i \in R$). Then $n_1 a_1 + \dots + n_k a_k \in P^r M$ for every positive integer r , whence each $\langle n_i \rangle \subseteq P^r \forall i$, for every power of P . Hence $\langle n_i \rangle \subseteq \bigcap_r P^r = 0$.¹ Thus $\langle n_i \rangle = 0$ and hence $n_i = 0$. Therefore, the arising contradiction establishes the statement.

Definition 3: A submodule N of an R -module M is called P -pure, if

$$P^r N = N \cap P^r M, \quad \text{for } r=1, 2, \dots$$

where P is a fixed prime ideal of the ring R .

Definition 4: Let M be an R -module. If $0 \neq a \in M$ and $O(a) = P^k$, where P is a prime ideal, then the element a is said to be of P -power order.

Lemma 2. *The submodule N generated by a P -independent subset $\{a_\lambda\}_{\lambda \in A}$ of M is P -pure in M . Conversely, if an independent set containing but elements of P -power order and/or of order 0 generates a P -pure submodule, then it is P -independent.*

PROOF. Suppose that $h \in N \cap P^r M$, where N is the submodule generated by a P -independent set $\{a_\lambda\}_{\lambda \in A}$ of M . Then $h = n_1 a_1 + \dots + n_k a_k \in P^r M$, where we may assume that $n_i a_i \neq 0$ ($n_i \in R, \forall i$). By P -independence there exist ideals B_i such that $\langle n_i \rangle = P^r B_i$ ($i=1, 2, \dots, k$). Let $n_i = \sum_j \alpha_j \beta_{ij}$ ($i=1, 2, \dots, k, \alpha_j \in P^r, \beta_{ij} \in B_i$). Now

$$h = \sum_{i=1}^k n_i a_i = \sum_i \left(\sum_j \alpha_j \beta_{ij} \right) a_i = \sum_j \alpha_j \left(\sum_i \beta_{ij} a_i \right) \in P^r N.$$

i.e. $P^r N = N \cap P^r M$. Hence N is P -pure in M .

Conversely, let $\{a_\lambda\}_{\lambda \in A}$ be an independent set such that a_λ are of P -power order and/or of order 0. Let $N = \langle \dots, a_\lambda, \dots \rangle$ is P -pure in M ; to show that $\{a_\lambda\}_{\lambda \in A}$ is P -independent.

¹ See ZARISKI and SAMUEL: Commutative Algebra I, pp. 216, Corollary 1.

Let $h = n_1 a_1 + \dots + n_k a_k \in P^r M$ ($n_i a_i \neq 0$, $n_i \in R$). By P -purity of N in M , we have

$n_1 a_1 + \dots + n_k a_k \in P^r N$, suppose

$$n_1 a_1 + \dots + n_k a_k = \sum_i p_i b_i \quad (p_i \in P^r, b_i \in N)$$

where,

$$b_i = n_{i_1} a_1 + n_{i_2} a_2 + \dots + n_{i_k} a_k \quad (n_{i_j} \in R, j=1, 2, \dots, k).$$

Hence,

$$\begin{aligned} n_1 a_1 + \dots + n_k a_k &= \sum_i p_i (n_{i_1} a_1 + \dots + n_{i_k} a_k) = \\ &= \sum_i p_i n_{i_1} a_1 + \sum_i p_i n_{i_2} a_2 + \dots + \sum_i p_i n_{i_k} a_k. \end{aligned}$$

By independence of $\{a_\lambda\}_{\lambda \in A}$, we have

$$n_j a_j = \sum_i p_i n_{i_j} a_j.$$

If $O(a_j) = 0$, then $n_j - \sum_i p_i n_{i_j} = 0$, hence

$$\langle n_j \rangle \subseteq P^r, \text{ i.e. } \{a_\lambda\}_{\lambda \in A} \text{ is } P\text{-independent.}$$

If $O(a_j)$ is P -power, then

$$n_j - \sum_i p_i n_{i_j} \in P^r, \text{ hence}$$

$$\langle n_j \rangle \subseteq P^r.$$

This proves the Lemma.

Definition 5: We call an R -module M P -divisible if it satisfies $PM = M$, where P is a fixed prime ideal of the ring R .

Definition 6: A submodule B of an R -module M will be called a P -basic submodule of M , if

- (i) B is the direct sum of cyclic P -modules and/or cyclic submodules of order 0,
- (ii) B is a P -pure submodule of M ,
- (iii) the factor module M/B is P -divisible.

[We emphasize that P denotes the same fixed prime ideal throughout.]

Remark. M is a P -basic submodule of itself if and only if (i) holds for M ; and 0 is a P -basic submodule of M if and only if M is P -divisible.

If B is a P -basic submodule of an R -module M then B may be written as $B = \bigoplus_{n=0}^{\infty} B_n$, where B_0 is free and B_n ($n \neq 0$) is a direct sum of cyclic P -modules of order ideal P^n .

Lemma 3. A subset $\{a_\lambda\}_{\lambda \in A}$ of an R -module M generates a P -basic submodule B of M if and only if it is a P -independent set containing but elements of P -power order and/or of order 0 and $M = \langle B, P^n M \rangle$ for each n .

PROOF. Let $\{a_\lambda\}_{\lambda \in A}$ be a P -independent set in M , then by Lemma 2, it generates a P -pure submodule B of M . Since $\{a_\lambda\}_{\lambda \in A}$ is P -independent, by Lemma 1, it is independent hence and by the given condition on a_λ 's. B is a direct sum of cyclic submodules, $B = \bigoplus_{n=0}^{\infty} B_k$, where we may suppose that B_0 is free and B_k is a direct sum of cyclic modules of order ideal P^k for $k=1, 2, \dots$. Now to show M/B is P -divisible. Since $M = \langle B, P^n M \rangle$ for each n , any $g \in M$, $g \neq 0$ can be written as

$$g = b + \sum_i \alpha_i g_i$$

where $b \in B$, $\alpha_i \in P^n$, $g_i \in M$. This implies that

$$g + B = \sum_i \alpha_i (g_i + B), \text{ i.e.,}$$

M/B is P -divisible.

Conversely, suppose $\{a_\lambda\}_{\lambda \in A}$ generates a P -basic submodule B of M . Then by Lemma 2 and from conditions (i) and (ii) of definition 6, we obtain the P -independence of the set $\{a_\lambda\}_{\lambda \in A}$. Since M/B is P -divisible, for $g \in M$, ($g \neq 0$) we have $g + B = \sum_i \alpha_i (g_i + B)$ ($\alpha_i \in P^n$, $g_i \in M$). This implies $\sum_i \alpha_i g_i - g \in B$, i.e. $\sum_i \alpha_i g_i - g = b$ for some $b \in B$. Hence $g = b + \sum_i \alpha_i g_i$. Thus $g \in \langle B, P^n M \rangle$, i.e., $M = \langle B, P^n M \rangle$ for every n .

Theorem 4. An R -module M contains P -basic submodules B for every prime ideal P of R if $M = \langle B, P^n M \rangle$ for each n .

PROOF. Every R -module M contains P -independent set $\{a_\lambda\}_{\lambda \in A}$ containing but elements of P -power order and/or of zero order. Then by given condition and using Lemma 3. $\{a_\lambda\}_{\lambda \in A}$ generates a P -basic submodule.

References

- [1] D. L. BOYER, On the theory of p -basic subgroups of abelian groups. Topics in abelian groups, Chicago, (1963), 323—330.
- [2] CH. W. CURTIS and I. REINER, Representation theory of finite groups and associative algebras. Interscience. New York (1962).
- [3] L. FUCHS, Notes on abelian group II. Acta Math. Acad. Sci. Hung. **11** (1960), 117—125.
- [4] L. FUCHS, Ranks of modules. Ann. Univ. Sci. Budapest. **6** (1963), 71—78.
- [5] L. FUCHS, Infinite abelian groups. Vol. I. Academic Press. New York and London (1970).

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