

The upper radical construction

By PAUL O. ENERSEN and W. G. LEAVITT (Lincoln, Nebraska)

Let $\mathbf{W}$ be a universal class of not necessarily associative rings. For a nonempty subclass $\mathbf{M}$ of $\mathbf{W}$ we consider the class function $\mathcal{U}\mathbf{M} = \{\mathbf{R} \in \mathbf{W} \mid \text{every } 0 \neq \mathbf{R}/\mathbf{I} \notin \mathbf{M}\}$. The class $\mathbf{M}$ is said to be *s-complete* if it satisfies the property: If $\mathbf{R} \in \mathbf{M}$, then for every $0 \neq \mathbf{I} \triangleleft \mathbf{R}$ there exists some $0 \neq \mathbf{I}/\mathbf{J} \in \mathbf{M}$ (we use the notation $\mathbf{I} \triangleleft \mathbf{R}$ to mean $\mathbf{I}$ is an ideal of $\mathbf{R}$). It is well known (see e.g. [2, pp. 6—7]) that if $\mathbf{M}$ is *s-complete*, $\mathcal{U}\mathbf{M}$ is a radical class called the *upper radical class* determined by $\mathbf{M}$. However, the converse fails as shown in Example 1. In Theorem 1 we give necessary and sufficient conditions on a class $\mathbf{M}$ in order that $\mathcal{U}\mathbf{M}$ be a radical class, a hereditary radical class, or a hypernilpotent radical class. In the remainder of this note we consider subclasses which determine the same upper radical class.

Recall that a nonempty subclass $\mathbf{P}$ of $\mathbf{W}$ is a radical class in $\mathbf{W}$ if it has the properties (i) $\mathbf{P}$ is homomorphically closed, and (ii) if $\mathbf{R} \in \mathbf{W}$ with $\mathbf{R} \notin \mathbf{P}$, then there exists $\mathbf{I} \triangleleft \mathbf{R}$ such that $0 \neq \mathbf{R}/\mathbf{I} \in \mathcal{S}\mathbf{P} = \{\mathbf{R} \in \mathbf{W} \mid \text{if } 0 \neq \mathbf{I} \triangleleft \mathbf{R} \text{ then } \mathbf{I} \notin \mathbf{P}\}$. $\mathbf{P}$ is said to be *hereditary* if whenever $\mathbf{R} \in \mathbf{P}$ then $\mathbf{I} \in \mathbf{P}$ for all $\mathbf{I} \triangleleft \mathbf{R}$. A hereditary radical class is called a *hypernilpotent* radical class if it contains all nilpotent rings.

Consider the following properties for a nonempty subclass $\mathbf{M}$ of $\mathbf{W}$:

- (A) If $0 \neq \mathbf{R} \in \mathbf{M}$, there exists some $0 \neq \mathbf{R}/\mathbf{I} \in \mathcal{S}\mathcal{U}\mathbf{M}$.
- (B) If $0 \neq \mathbf{I} \triangleleft \mathbf{R} \in \mathbf{W}$ is such that some $0 \neq \mathbf{I}/\mathbf{J} \in \mathbf{M}$, there exists $\mathbf{H} \triangleleft \mathbf{R}$ such that $0 \neq \mathbf{R}/\mathbf{H} \in \mathcal{S}\mathcal{U}\mathbf{M}$.
- (C) If $0 \neq \mathbf{R} \in \mathbf{W}$ is such that $\mathbf{R}^2 = 0$, then $\mathbf{R} \notin \mathbf{M}$.

Theorem 1. *The class $\mathcal{U}\mathbf{M}$ is a radical class in $\mathbf{W}$ if and only if $\mathbf{M}$ has property (A). $\mathcal{U}\mathbf{M}$ is a hereditary radical class if and only if $\mathbf{M}$ has properties (A) and (B), and is a hypernilpotent radical class if and only if $\mathbf{M}$ has properties (A), (B), and (C).*

PROOF. First note that for any class $\mathbf{M}$, $\mathcal{U}\mathbf{M}$ is always homomorphically closed. Suppose $\mathbf{M}$ satisfies property (A) and let $0 \neq \mathbf{R} \in \mathbf{W}$ with $\mathbf{R} \notin \mathcal{U}\mathbf{M}$. Then there is some $0 \neq \mathbf{R}/\mathbf{I} \in \mathbf{M}$, so by (A) there exists $0 \neq (\mathbf{R}/\mathbf{I})/(\mathbf{K}/\mathbf{I}) \cong \mathbf{R}/\mathbf{K} \in \mathcal{S}\mathcal{U}\mathbf{M}$. Thus $\mathcal{U}\mathbf{M}$ is radical.

Conversely, suppose $\mathcal{U}\mathbf{M}$ is radical. If $0 \neq \mathbf{R} \in \mathbf{M}$, then since $\mathbf{M} \cap \mathcal{U}\mathbf{M} = \{0\}$, $\mathbf{R} \notin \mathcal{U}\mathbf{M}$. Thus by (ii) there exists some $0 \neq \mathbf{R}/\mathbf{I} \in \mathcal{S}\mathcal{U}\mathbf{M}$.

To see the equivalence of hereditariness and property (B) in the case $\mathcal{U}\mathbf{M}$ is a radical class, first suppose $\mathbf{M}$ has property (B). Let $0 \neq \mathbf{R} \in \mathcal{U}\mathbf{M}$ and consider any $0 \neq \mathbf{I} \triangleleft \mathbf{R}$. If $\mathbf{I} \notin \mathcal{U}\mathbf{M}$, there exists $0 \neq \mathbf{I}/\mathbf{J} \in \mathbf{M}$, so by (B) there exists $0 \neq \mathbf{R}/\mathbf{H} \in \mathcal{S}\mathcal{U}\mathbf{M}$, contradicting (i). Thus $\mathcal{U}\mathbf{M}$ is hereditary.

On the other hand suppose $\mathcal{U}\mathbf{M}$ is hereditary. Let $0 \neq \mathbf{R} \in \mathbf{W}$ and let $0 \neq \mathbf{I} \triangleleft \mathbf{R}$ be such that some $0 \neq \mathbf{I}/\mathbf{J} \in \mathbf{M}$. Then $\mathbf{I} \notin \mathcal{U}\mathbf{M}$, so by the hereditariness of $\mathcal{U}\mathbf{M}$, $\mathbf{R} \notin \mathcal{U}\mathbf{M}$. Thus by (ii), $0 \neq \mathbf{R}/\mathbf{H} \in \mathcal{S}\mathcal{U}\mathbf{M}$ for some $\mathbf{H} \triangleleft \mathbf{R}$.

Clearly if $\mathcal{UM}$ is a hypernilpotent radical class then $\mathbf{M}$ satisfies property (C). To see the converse, let $0 \neq \mathbf{R} \in \mathbf{W}$ be nilpotent and suppose $\mathbf{R} \notin \mathcal{UM}$. Then there exists $\mathbf{I} \triangleleft \mathbf{R}$ such that $0 \neq \mathbf{R}/\mathbf{I} \in \mathcal{S}\mathcal{UM}$. Since $\mathbf{R}$ is nilpotent, $\mathbf{R}/\mathbf{I}$ is also with index of nilpotency $m > 1$. Let $\mathbf{K}/\mathbf{I} = (\mathbf{R}/\mathbf{I})^{m-1}$. Then $0 \neq \mathbf{K}/\mathbf{I} \triangleleft \mathbf{R}/\mathbf{I}$ and $(\mathbf{K}/\mathbf{I})^2 = 0$. Furthermore, any image of $\mathbf{K}/\mathbf{I}$ must also have zero multiplication. Thus by (C), no nonzero image of $\mathbf{K}/\mathbf{I}$ lies in $\mathbf{M}$, contradicting $\mathbf{R}/\mathbf{I} \in \mathcal{S}\mathcal{UM}$. Therefore $\mathbf{R} \in \mathcal{UM}$, and so $\mathcal{UM}$ is hypernilpotent.

Example 1: Let $\mathbf{N}$ denote the class of all nil rings. Then $\mathbf{N}$ is a radical class in the universal class $\mathbf{W}$ of all not necessarily associative rings (see e.g. [2, pp. 18–19]) with semisimple class $\mathcal{SN} = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R} \text{ has no nil ideals}\}$. Let $\mathbf{K}$ be any non-nil ring which has a nonzero nil ideal $\mathbf{J}$ (e.g., take $\mathbf{K} = \mathbf{F}[x]/(x)^2$, $\mathbf{F}$ any field). Let $\mathbf{M} = \mathcal{SN} \cup \{\mathbf{K}\}$. Then $\mathbf{M}$ is not s -complete, for every image of $\mathbf{J}$ must also be a nil ring and hence not in $\mathbf{M}$.

Let $0 \neq \mathbf{R} \in \mathbf{M}$, so that either $\mathbf{R} \in \mathcal{SN}$ or $\mathbf{R} \cong \mathbf{K}$.

Case 1: If $\mathbf{R} \in \mathcal{SN}$, $\mathbf{R}$ has no nil ideals. Thus if $\mathbf{H}$ is any nonzero ideal of $\mathbf{R}$, $\mathbf{N}(\mathbf{H}) \neq \mathbf{H}$, where $\mathbf{N}(\mathbf{H})$ is the nil radical of $\mathbf{H}$. Hence $0 \neq \mathbf{H}/\mathbf{N}(\mathbf{H}) \in \mathcal{SN} \subseteq \mathbf{M}$. Therefore every nonzero ideal of $\mathbf{R}$ has a nonzero image in $\mathbf{M}$; i.e., $\mathbf{R} \in \mathcal{S}\mathcal{UM}$.

Case 2: If $\mathbf{R} \cong \mathbf{K}$, then $\mathbf{N}(\mathbf{R})$ is a proper ideal of $\mathbf{R}$. Thus $0 \neq \mathbf{R}/\mathbf{N}(\mathbf{R}) \in \mathcal{SN}$, and by Case 1 we see that $\mathbf{R}/\mathbf{N}(\mathbf{R}) \in \mathcal{S}\mathcal{UM}$.

Therefore by Theorem 1, $\mathcal{UM}$ is a radical class.

We give one further application of Theorem 1.

Example 2: A ring $\mathbf{K}$ is called *completely idempotent* if $(a^2) = (a)$ for all $a \in \mathbf{K}$. In a recent paper [1] ANDRUNAKIEVITCH and RJABUHIN proved the following:

Theorem. *The following conditions are equivalent for an associative ring $\mathbf{K}$:*

- (1) $\mathbf{K}$ is completely idempotent,
- (2) $\mathbf{K}/\mathbf{I}$ has no nilpotent elements for all ideals $\mathbf{I}$ of $\mathbf{K}$,
- (3) If $\mathbf{K}/\mathbf{I}$ is subdirectly irreducible it has no zero divisors, and
- (4) Every $\mathbf{K}/\mathbf{I}$ is a subdirect sum of subdirectly irreducible rings with no zero divisors.

Either of the conditions (2) or (3) can be used to provide another proof that the class of all completely idempotent rings is a radical class in the universal class $\mathbf{W}$ of all associative rings:

First let $\mathbf{M} = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R} \text{ contains a nonzero nilpotent element}\}$ and let $\mathbf{K} \in \mathbf{M}$. Pick $0 \neq a \in \mathbf{K}$ to be a nilpotent. Let $\mathbf{I} \triangleleft \mathbf{K}$ be maximal with respect to exclusion of a . Then $\mathbf{K}/\mathbf{I}$ is subdirectly irreducible with nonzero heart, namely the principal ideal generated by the image of a in $\mathbf{K}/\mathbf{I}$. Since every ideal of $\mathbf{K}/\mathbf{I}$ contains a nonzero nilpotent, every ideal is in $\mathbf{M}$ and thus not in $\mathcal{UM}$. Hence $\mathbf{K}/\mathbf{I} \in \mathcal{S}\mathcal{UM}$. Therefore from Theorem 1 it follows that $\mathcal{UM}$, the class of all completely idempotent rings by (2), is a radical class.

To apply (3), let $\mathbf{M}' = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R} \text{ is a subdirectly irreducible ring containing a zero divisor}\}$ and let $\mathbf{K} \in \mathbf{M}'$. If $\mathbf{K}$ is completely idempotent, then [1, Theorem 1, p. 1015] $bc = 0$ for $b, c \in \mathbf{K}$ implies that $(b) \cap (c) = (0)$, contradicting the subdirect irreducibility of $\mathbf{K}$. Hence for some $a \in \mathbf{K}$, $(a^2) \neq (a)$. So if $\mathbf{I} \triangleleft \mathbf{K}$ is chosen to be maximal

with respect to $(a^2) \subseteq I$ and $a \notin I$, then K/I is subdirectly irreducible with heart $(\bar{a})$, where $\bar{a}$ is the image of a in K/I . For any $0 \neq J \triangleleft K/I$, $\bar{a} \in J$ so J has an ideal H maximal with respect to $\bar{a} \notin H$. Then J/H contains a nonzero nilpotent element and thus is in M' . Hence $J \notin \mathcal{UM}'$, and so $K/I \in \mathcal{S}\mathcal{UM}'$. So again, $\mathcal{UM}'$, the class of all completely idempotent rings by (3), is seen to be a radical class.

We now consider certain subclasses M and N of the arbitrary universal class W for which $\mathcal{UM} = \mathcal{UN}$. It should be noted that the results hold whether or not $\mathcal{UM}$ and $\mathcal{UN}$ are radical classes.

Theorem 2. *Let M and N be nonempty subclasses of W such that one of M and N is homomorphically closed. Then $\mathcal{UM} = \mathcal{UN}$ if and only if M and N satisfy property (a): Every nonzero ring in $M \cup N$ has a nonzero image in $M \cap N$.*

PROOF. Without loss of generality assume that M is the homomorphically closed class.

Suppose that $\mathcal{UM} = \mathcal{UN}$ and let $0 \neq R \in M \cup N$. If $R \in M$, then $R \notin \mathcal{UM} = \mathcal{UN}$ so there exists $0 \neq R/I \in N$. Thus $R/I \in M \cap N$. If $R \in N$, then $R \notin \mathcal{UM}$ so there exists $0 \neq R/I \in M$. Now $R/I \notin \mathcal{UN}$, so there exists $K \triangleleft R$, $I \subseteq K$, such that $0 \neq (R/I)/(K/I) \cong R/K \in N$. Since R/K is a homomorphic image of R/I , $R/K \in M \cap N$.

Conversely suppose $R \notin \mathcal{UM}$. Then there exists $0 \neq R/I \in M$. By property (a), we can find $K \triangleleft R$, $I \subseteq K$, such that $(R/I)/(K/I) \cong R/K \in M \cap N \subseteq N$. Hence $R \notin \mathcal{UN}$ and so $\mathcal{UN} \subseteq \mathcal{UM}$. Similarly $\mathcal{UM} \subseteq \mathcal{UN}$, and we have the desired equality.

Corollary 1: Property (a) is a sufficient condition that $\mathcal{UM} = \mathcal{UN}$ for arbitrary classes M and N .

Theorem 3. *If M and N are subclasses of W , one of which consists entirely of Noetherian rings, then $\mathcal{UM} = \mathcal{UN}$ if and only if M and N satisfy property (a).*

PROOF. Without loss of generality assume that M is the Noetherian class.

Suppose $\mathcal{UM} = \mathcal{UN}$ and let $0 \neq R \in M \cup N$. If $R \in M$, then $R \notin \mathcal{UN}$ so there exists $I_1 \triangleleft R$ such that $0 \neq R/I_1 \in N$. Then $R/I_1 \notin \mathcal{UM}$ so there exists $I_2 \triangleleft R$, $I_1 \subseteq I_2$, such that $0 \neq (R/I_1)/(I_2/I_1) \cong R/I_2 \in M$. Continuing the process we obtain an ascending chain of ideals of R , $(0) = I_0 \subseteq I_1 \subseteq I_2 \subseteq \dots$, so since R is Noetherian $I_n = I_{n+1}$ for some $n \geq 0$. Then $R/I_n = R/I_{n+1} \in M \cap N$.

If $R \in N$, as above we get $0 \neq R/J_1 \in M$. Then in the same manner we arrive at an ascending chain of ideals of R/J_1 , $J_1/J_1 \subseteq J_2/J_1 \subseteq J_3/J_1 \subseteq \dots$ which must stop at some $n \geq 1$. Then $R/J_n \cong (R/J_1)/(J_n/J_1) = (R/J_1)/(J_{n+1}/J_1) \cong R/J_{n+1} \in M \cap N$. Therefore property (a) is established and Corollary 1 yields the converse.

Example 3: Let $\{x_i\}$ $i=1, 2, \dots$ be a countably infinite set of indeterminates over Z , the ring of integers, and let $K = Z[x_1, x_2, \dots]$. For each $n=1, 2, \dots$ let $I_n = (x_1, 2x_2, \dots, nx_n)$ and define $K_n = K/I_n$. Let W be any universal class containing K and define subclasses M and N of W by $M = \{0, K_2, K_4, \dots, K_{2n}, \dots\}$ and $N = \{0, K_1, K_3, \dots, K_{2n-1}, \dots\}$. Note that for each n , K_n is not Noetherian, and that if $n > m$, then K_n is a homomorphic image of K_m .

Now let $R \in \mathcal{UM}$ and suppose that $R \notin \mathcal{UN}$. Then there exists $I \triangleleft R$ such that $0 \neq R/I \in N$; say $R/I \cong K_{2n-1}$. But then K_{2n} is a homomorphic image of R/I , and thus of R , which is in M , a contradiction. Hence $\mathcal{UM} \subseteq \mathcal{UN}$. The reverse inclusion follows similarly, so $\mathcal{UM} = \mathcal{UN}$.

However, $M \cap N = \{0\}$, so property (a) does not hold for M and N . Thus we see that the hypotheses that one class be homomorphically closed, and that one class be Noetherian, cannot be dropped from Theorems 2 and 3, respectively.

As one application of Corollary 1, we have:

Theorem 4. *Let M be a nonempty subclass of W . Then $\mathcal{U}(W - \mathcal{U}M) = \mathcal{U}M$.*

PROOF. For any $N \subseteq W$, $\mathcal{U}N = \mathcal{U}(N \cup \{0\})$. Thus it suffices to show that $\mathcal{U}M' = \mathcal{U}M$, where $M' = (W - \mathcal{U}M) \cup \{0\}$.

Let $0 \neq R \in M'$. Then $R \notin \mathcal{U}M$, so there exists $0 \neq R/I \in M$. Since $M \subseteq M'$, the result follows from Corollary 1.

Noting that $M \subseteq N$ implies $\mathcal{U}N \subseteq \mathcal{U}M$, we have:

Corollary 2: *If M is a subclass of W , then $\mathcal{U}M = \mathcal{U}N$ for any subclass N such that $M \subseteq N \subseteq (W - \mathcal{U}M) \cup \{0\}$.*

References

- [1] B. A. ANDRUNAKIEVITCH and JU. M. RJABUHIN, Completely idempotent rings and the generalized nil radical, *Dokl. Akad. Nauk. SSSR* **201** (1971), 1015—1018. (*Russian*)
- [2] N. J. DIVINSKY, Rings and radicals, *Toronto, (Ont.)*, 1965.

(Received 17. June 1972.)