

Rearrangement inequalities

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1. Introduction

Let $x = (x_1, x_2, \dots, x_n)$ be an n -tuple of real numbers, where $x_i \in (a, b)$, $i = 1, 2, \dots, n$ and $-\infty \leq a < b \leq \infty$. Let us denote by $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ the same n -tuple rearranged in a nonincreasing order $x_1^* \geq x_2^* \geq \dots \geq x_n^*$ and $x' = (x'_1, x'_2, \dots, x'_n)$ the same n -tuple rearranged in a nondecreasing order $x'_1 \leq x'_2 \leq \dots \leq x'_n$.

Let $F(x) = F(x_1, x_2, \dots, x_n)$ ($x_i \in (a, b)$, $i = 1, 2, \dots, n$) be a real valued function. We say that the function F has the property $\mathcal{P}$ if for any $x = (x_1, x_2, \dots, x_n)$ ($x_i \in (a, b)$, $i = 1, 2, \dots, n$) the inequality

$$(1.1) \quad F(x') \leq F(x) \leq F(x^*)$$

holds.

For a function F with property $\mathcal{P}$ an example is the following one. Suppose that $(a, b) = (0, \infty)$ and $0 \leq a_1 \leq a_2 \leq \dots \leq a_n$ is a given sequence. Then the function

$$(1.2) \quad F(x) = \sum_{i=1}^n \frac{a_i}{x_i}$$

has property $\mathcal{P}$ (see [1], Theorem 368).

Recently, LONDON [3] proved — as a generalization of (1.2) — that the functions of the form

$$(1.3) \quad F(x) = \sum_{i=1}^n f\left(\frac{a_i}{x_i}\right) \quad (0 \leq a_1 \leq a_2 \leq \dots \leq a_n)$$

have the property $\mathcal{P}$ if $f(t)$ is a convex function for $t \geq 0$ and satisfies $f(0) \leq f(t)$ for any $t \geq 0$. In the same paper one can find — as a result independent from the former one — a sufficient condition for the $\mathcal{P}$ property of the functions of the type

$$(1.4) \quad F(x) = \sum_{i=1}^n f\left(1 + \frac{a_i}{x_i}\right) \quad (0 \leq a_1 \leq a_2 \leq \dots \leq a_n).$$

This result is a generalization of a theorem given by MINC [4].

In the present note as a common generalization of the earlier results we give a necessary and sufficient condition for the $\mathcal{P}$ property of the functions having the form

$$(1.5) \quad F(x) = \sum_{i=1}^n g_i(x_i),$$

where $g_i(t)$, $i=1, 2, \dots, n$, $t \in (a, b)$ are given real valued functions. Further, we also give a sufficient condition for the inequality

$$(1.6) \quad \sum_{i=1}^n \sum_{k=1}^n g_{i+k}(x_i y_k) \leq \sum_{i=1}^n \sum_{k=1}^n g_{i+k}(x_i^* y_k^*),$$

which follows from our main theorem and can be considered as a generalization of a result of WIENER [5] (see also [1], Theorem 386).

2. The main theorem

Let $g_i(t)$, $i=1, 2, \dots, n$ be real valued functions defined in the interval (a, b) .

Theorem 1. *The function*

$$(2.1) \quad F(x) = \sum_{i=1}^n g_i(x_i) \quad (x_i \in (a, b), i=1, 2, \dots, n)$$

has the property $\mathcal{P}$ if and only if the functions

$$(2.2) \quad g_k(t) - g_{k+1}(t) \quad k = 1, 2, \dots, n-1$$

are nondecreasing in the interval (a, b) .

PROOF. (i) The condition is necessary. From the property $\mathcal{P}$ it follows

$$(2.3) \quad \sum_{i=1}^n g_i(x'_i) \leq \sum_{i=1}^n g_i(x_i)$$

for any $x = (x_1, x_2, \dots, x_n)$ ($x_i \in (a, b)$, $i=1, 2, \dots, n$). Let $1 \leq k \leq n-1$ and $a < t_1 \leq t_2 < b$ be arbitrary variables and let the n -tuple

$$x = (\underbrace{1}_{t_1}, \underbrace{2}_{t_1}, \dots, \underbrace{k-1}_{t_1}, \underbrace{k}_{t_2}, \underbrace{k+1}_{t_1}, \underbrace{k+2}_{t_2}, \dots, \underbrace{n}_{t_2})$$

be substituted into (2.3). Then we have

$$g_k(t_1) + g_{k+1}(t_2) \leq g_k(t_2) + g_{k+1}(t_1),$$

i.e. the function $g_k(t) - g_{k+1}(t)$ is nondecreasing.

(ii) The condition is sufficient. First, it should be noted that the function $g_l - g_j$ is nondecreasing if $1 \leq l < j \leq n$, since it can be expressed as the sum of nondecreasing functions:

$$g_l - g_j = \sum_{k=l}^{j-1} (g_k - g_{k+1}).$$

This means that the inequality

$$(2.4) \quad g_l(t) + g_j(s) \leq g_l(s) + g_j(t)$$

is true for all $1 \leq l < j \leq n$ and $a < t \leq s < b$. Let us now consider an arbitrary n -tuple

$$x = (x_1, x_2, \dots, \overset{l}{x_l}, \dots, \overset{j}{x_j}, \dots, x_n) \quad (1 \leq l < j \leq n).$$

We introduce the transformation A_{lj} by

$$A_{lj}x = (x_1, x_2, \dots, \overset{l}{x_j}, \dots, \overset{j}{x_l}, \dots, x_n) \quad (l < j)$$

which means an interchange of the l -th and j -th elements in x . If $l < j$ and $x_l \geq x_j$, then by (2.4) we obtain

$$g_l(x_j) + g_j(x_l) \leq g_l(x_l) + g_j(x_j),$$

hence

$$(2.5) \quad F(A_{lj}x) = \sum_{\substack{i=1 \\ i \neq l, j}}^n g_i(x_i) + g_l(x_j) + g_j(x_l) \leq \sum_{\substack{i=1 \\ i \neq l, j}}^n g_i(x_i) + g_l(x_l) + g_j(x_j) = F(x).$$

The n -tuple x' can be obtained from x by applying a product of transformations of the type A_{lj} ($l < j$), while $x_l \geq x_j$, therefore by (2.5) we have

$$F(x') \leq F(x).$$

The other inequality can similarly be proved because

$$F(x) \leq F(A_{lj}x)$$

for $l < j$ and $x_l \leq x_j$, and x^* can be expressed as consecutive A_{lj} ($l < j$) transformations performed on x , while $x_l \geq x_j$. Therefore

$$F(x) \leq F(x^*).$$

Remark. The factorization used in the proof of the theorem can be verified by induction. The statement is clearly true for $n=2$. Let us suppose that the statement is also true for $n=k \geq 2$ and let $x = (x_1, x_2, \dots, x_k, x_{k+1})$ be an arbitrary $(k+1)$ -tuple. If $x'_1 = x_1$, then the proof is completed, because for the k -tuple $(x_2, x_3, \dots, x_k, x_{k+1})$ the statement is true which implies that it is true for x . If $x'_1 = x_j$ ($1 < j$), then by definition of x' we have $x_1 \geq x_j$ and thus the $(k+1)$ -tuples $A_{1j}x$ and x' will have the same first element. For the further k elements of the $(k+1)$ -tuple $A_{1j}x$ the statement will be true by assumption.

For example, if $x = (1, 5, 7, 2, 3, 4, 6)$, then we have

$$x' = A_{67}A_{56}A_{46}A_{35}A_{24}x$$

and similarly

$$x^* = A_{46}A_{37}A_{27}A_{13}x.$$

3. Special cases

(a) First, we consider the functions having the form

$$(3.1) \quad F(x) = \sum_{i=1}^n \frac{a_i}{x_i} \quad (x_i \in (0, \infty), i = 1, 2, \dots, n),$$

where $a_i \geq 0$, $i = 1, 2, \dots, n$ are given numbers. By Theorem 1 the function (3. 1) has the $\mathcal{P}$ property if and only if the functions

$$\frac{a_k}{t} - \frac{a_{k+1}}{t} = \frac{a_k - a_{k+1}}{t} \quad k = 1, 2, \dots, n-1$$

are nondecreasing, i.e. $a_k \leq a_{k+1}$, $k = 1, 2, \dots, n-1$. This means that (3. 1) has the property $\mathcal{P}$ if and only if the sequence $\{a_k\}_1^n$ is nondecreasing. This is a generalization of the theorem 368 in [1] because we also proved the necessity of the condition. We remark that this result is valid for all real, not necessarily nonnegative a_i , $i = 1, 2, \dots, n$.

(b) As a generalization of the example (a) we consider the functions

$$(3. 2) \quad F(x) = \sum_{i=1}^n f\left(\frac{a_i}{x_i}\right) \quad (x_i \in (0, \infty), i = 1, 2, \dots, n),$$

where $a_i \geq 0$, $i = 1, 2, \dots, n$ are given numbers and $f(t)$ is a real valued function in the interval $[0, \infty)$. By Theorem 1 the function (3. 2) has the property $\mathcal{P}$ if and only if the functions

$$(3. 3) \quad \varphi_k(t) = f\left(\frac{a_k}{t}\right) - f\left(\frac{a_{k+1}}{t}\right) \quad k = 1, 2, \dots, n-1$$

are nondecreasing in the interval $(0, \infty)$. By reason of this result Theorem 2 of LONDON's paper [3] states the following.

Theorem 2. *If $f(t)$ is convex in the interval $[0, \infty)$, $f(0) \leq f(t)$ for $t \geq 0$ and $0 \leq a_k \leq a_{k+1}$, then the function (3. 3) is nondecreasing.*

The proof of this theorem will be performed in a way considerable different as given in [3]. For this the following simple Lemma will be used.

Lemma. *If $f(t)$ is convex in the interval $[0, \infty)$ and $f(0) \leq f(t)$ for $t \geq 0$, then*

$$(3. 4) \quad f(x_1) + f(x_2) \leq f(y_1) + f(y_2)$$

for arbitrary $x_i, y_i \in [0, \infty)$ with the property $x_1 \leq y_1$ and $x_1 + x_2 \leq y_1 + y_2$.

PROOF. Let $x_3 = y_1 + y_2 - x_1 - x_2$ and $y_3 = 0$. From the well known theorem of KARAMATA [2] (see also [1], Theorem 108) we have

$$f(x_1) + f(x_2) + f(x_3) \leq f(y_1) + f(y_2) + f(y_3),$$

hence, because of

$$f(y_3) = f(0) \leq f(y_1 + y_2 - x_1 - x_2) = f(x_3),$$

we obtain (3. 4).

PROOF of the Theorem 2. Let $0 < x < y$ be arbitrary numbers and we define

$$x_1 = \frac{a_k}{x}, \quad x_2 = \frac{a_{k+1}}{y}, \quad y_1 = \frac{a_{k+1}}{x}, \quad y_2 = \frac{a_k}{y}.$$

We have $x_1 \leq y_1$ and $x_1 + x_2 \leq y_1 + y_2$, which implies (3. 4) by the Lemma, i.e.

$$f\left(\frac{a_k}{x}\right) + f\left(\frac{a_{k+1}}{y}\right) \leq f\left(\frac{a_{k+1}}{x}\right) + f\left(\frac{a_k}{y}\right).$$

Hence

$$\varphi_k(x) \leq \varphi_k(y).$$

This means that φ_k is nondecreasing.

(c) We now consider the functions

$$(3.5) \quad F(x) = \sum_{i=1}^n \log \left(1 + \frac{a_i}{x_i} \right) \quad (x_i \in (0, \infty), i = 1, 2, \dots, n)$$

where $a_i \geq 0$, $i = 1, 2, \dots, n$ are given numbers. By Theorem 1 the function (3.5) has the property $\mathcal{P}$ if and only if the functions

$$\log \left(1 + \frac{a_k}{t} \right) - \log \left(1 + \frac{a_{k+1}}{t} \right) = \log \frac{t + a_k}{t + a_{k+1}} \quad k = 1, 2, \dots, n-1$$

are nondecreasing, i.e. the sequence $\{a_k\}_1^n$ is nondecreasing. This result is a generalization of the theorem of MINC [4]. We remark that this result is independent from Theorem 2 because the function $\log(1+t)$ is concave.

(d) As a generalization of the example (c) we consider the functions having the form

$$(3.6) \quad F(x) = \sum_{i=1}^n f \left(1 + \frac{a_i}{x_i} \right) \quad (x_i \in (0, \infty), i = 1, 2, \dots, n),$$

where $a_i \geq 0$, $i = 1, 2, \dots, n$ are given numbers and $f(t)$ is a real valued function defined in the interval $[1, \infty)$. By theorem 1 the function (3.6) has the property $\mathcal{P}$ if and only if the functions

$$(3.7) \quad \psi_k(t) = f \left(1 + \frac{a_k}{t} \right) - f \left(1 + \frac{a_{k+1}}{t} \right) \quad k = 1, 2, \dots, n-1$$

are nondecreasing in the interval $(0, \infty)$. By reason of this result Theorem 1 of [3] can be presented also in the following form.

Theorem 3. Let $f(t)$ be a real valued function in the interval $[0, \infty)$ for which

- (i) $f(1) \leq f(t)$ if $t \geq 1$; and
- (ii) $h(t) = f(e^t)$ is convex in $[0, \infty)$, are valid. Then the function (3.7) is non-decreasing provided that $0 \leq a_k \leq a_{k+1}$.

PROOF. We shall show that this result can easily be obtained from the Lemma, too. By the conditions (i) and (ii) the function $h(t)$ is convex in $[0, \infty)$ and $h(0) \leq h(t)$ for all $t \geq 0$. Let $0 < x < y$ be arbitrary numbers and we define

$$\begin{aligned} x_1 &= \log \left(1 + \frac{a_k}{x} \right), & x_2 &= \log \left(1 + \frac{a_{k+1}}{y} \right), \\ y_1 &= \log \left(1 + \frac{a_{k+1}}{x} \right), & y_2 &= \log \left(1 + \frac{a_k}{y} \right). \end{aligned}$$

We have $x_1 \leq y_1$ and $x_1 + x_2 \leq y_1 + y_2$, which — by the Lemma — implies

$$h(x_1) + h(x_2) \leq h(y_1) + h(y_2),$$

i.e.

$$h \left[\log \left(1 + \frac{a_k}{x} \right) \right] - h \left[\log \left(1 + \frac{a_{k+1}}{x} \right) \right] \leq h \left[\log \left(1 + \frac{a_k}{y} \right) \right] - h \left[\log \left(1 + \frac{a_{k+1}}{y} \right) \right].$$

From this inequality — because of $f(1+t) = h[\log(1+t)]$ — we obtain

$$\psi_k(x) \leq \psi_k(y),$$

i.e. the function ψ_k is nondecreasing.

4. On the Wiener's inequality

The inequality of WIENER [5] states that if $c_2 \geq c_3 \geq \dots \geq c_{2n} \geq 0$ and

$$x = (x_1, x_2, \dots, x_n), \quad y = (y_1, y_2, \dots, y_n) \quad (x_i \geq 0, y_i \geq 0, i = 1, 2, \dots, n)$$

are arbitrary n -tuples, then

$$(4.1) \quad \sum_{i=1}^n \sum_{k=1}^n c_{i+k} x_i y_k \leq \sum_{i=1}^n \sum_{k=1}^n c_{i+k} x_i^* y_k^*.$$

As a generalization of this theorem we shall prove the following one.

Theorem 4. Let $g_l(t)$, $l=2, 3, \dots, 2n$ be real valued functions defined in the interval $(0, \infty)$. For arbitrary n -tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ ($x_i, y_i \in (0, \infty)$, $i=1, 2, \dots, n$) the inequality

$$(4.2) \quad \sum_{i=1}^n \sum_{k=1}^n g_{i+k}(x_i y_k) \leq \sum_{i=1}^n \sum_{k=1}^n g_{i+k}(x_i^* y_k^*)$$

holds if the functions

$$(4.3) \quad g_l(t) - g_{l+1}(t) \quad l = 2, 3, \dots, 2n-1$$

are nondecreasing in the interval $(0, \infty)$.

PROOF. By Theorem 1 we have

$$\begin{aligned} \sum_{i=1}^n \sum_{k=1}^n g_{i+k}(x_i y_k) &= \sum_{i=1}^n \left(\sum_{k=1}^n g_{i+k}(x_i y_k) \right) \leq \\ &\leq \sum_{i=1}^n \left(\sum_{k=1}^n g_{i+k}(x_i y_k^*) \right) = \sum_{k=1}^n \left(\sum_{i=1}^n g_{i+k}(x_i y_k^*) \right) \leq \sum_{k=1}^n \left(\sum_{i=1}^n g_{i+k}(x_i^* y_k^*) \right), \end{aligned}$$

which proves the theorem.

As an example we consider the functions $g_l(t) = c_l t$, where c_l , $l=2, 3, \dots, 2n$ are given numbers. By Theorem 4 the inequality (4.1) holds if the sequence $\{c_l\}_{l=2}^{2n}$ is nonincreasing.

As a further example we define the functions

$$g_l(t) = \frac{c_l}{t} \log \left(1 + \frac{c_l}{t} \right) \quad l = 2, 3, \dots, 2n,$$

where $0 \leq c_2 \leq c_3 \leq \dots \leq c_{2n}$ are given numbers. By Theorem 4 we have the inequality

$$\sum_{i=1}^n \sum_{k=1}^n \frac{c_{i+k}}{x_i y_k} \log \left(1 + \frac{c_{i+k}}{x_i y_k} \right) \leq \sum_{i=1}^n \sum_{k=1}^n \frac{c_{i+k}}{x_i^* y_k^*} \log \left(1 + \frac{c_{i+k}}{x_i^* y_k^*} \right)$$

for all $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ ($x_i, y_i \in (0, \infty)$, $i = 1, 2, \dots, n$). This inequality is the same as the following one:

$$G(x, y) = \prod_{i=1}^n \prod_{k=1}^n \left(\frac{c_{i+k}}{x_i y_k} + 1 \right)^{\frac{c_{i+k}}{x_i y_k}} \leq G(x^*, y^*).$$

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