

On the behaviour of Laplace-integrals under conformal mappings of the region of convergence

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Let ζ be a complex number with $0 < |\zeta| < 1$ and by

$$\psi(z) = \frac{z - \zeta}{1 - \bar{\zeta}z}$$

denote a conformal mapping of the unit circle onto itself. TURÁN [9] proved

Theorem A: *Given any ζ with $0 < |\zeta| < 1$ there is a function*

$$f_1(z) = \sum_{v=0}^{\infty} a_v z^v$$

regular for $|z| < 1$ with convergent $\sum_{v=0}^{\infty} a_v$ and such that the series

$$f_2(z) = f_1(\psi(z)) = \sum_{v=0}^{\infty} b_v(\zeta) z^v$$

diverges for the corresponding

$$z = \psi^{-1}(1) = \frac{1 + \zeta}{1 + \bar{\zeta}}.$$

If $f_1(z) = \sum_{v=0}^{\infty} a_v z^v$ is convergent at $z=1$ then $f_2(z)$ is Abel-summable at $z=\psi^{-1}(1)$.

Several authors investigated similar problems (see e.g. the references in [5]). Especially ALPÁR [1] proved

Theorem B: *Given any ζ with $0 < |\zeta| < 1$ there is a function*

$$f_1(z) = \sum_{v=0}^{\infty} a_v z^v$$

regular for $|z| < 1$ with $\sum_{v=0}^{\infty} |a_v| < \infty$ and such that

$$f_2(z) = f_1(\psi(z)) = \sum_{v=0}^{\infty} b_v(\zeta) z^v$$

with

$$\sum_{v=0}^{\infty} |b_v(\zeta)| = \infty.$$

In this note we shall prove some similar results for *Laplace-integrals*.
Let $\alpha, \beta, \gamma, \delta$ be real numbers with

$$(1) \quad \alpha\delta - \beta\gamma = 1.$$

Then the right halfplane $\operatorname{Re} s > 0$ is mapped conformally onto itself by

$$\varphi(s) = \frac{\alpha s + i\beta}{\delta - i\gamma s}.$$

In the following we consider nontrivial conformal mappings, i.e. we assume

$$(2) \quad \alpha\gamma \neq 0.$$

Consider s with $\operatorname{Re} s > 0$ and set

$$f_1(s) = \int_0^{\infty} e^{-st} F_1(t) dt + K_1.$$

Then in general neither of the following relations hold for $\operatorname{Re} s > 0$:

$$f_2(s) = f_1(\varphi(s)) = \int_0^{\infty} e^{-st} F_2(t) dt + K_2$$

$$f_2(s) = f_1(\varphi(s)) = \int_0^{\infty} e^{-st} dF_2(t).$$

As an example take ($\operatorname{Re} s > 0$)

$$f_1(s) = \log \frac{s-i}{i(s-2i)} = \int_0^{\infty} e^{-st} \frac{e^{2it} - e^{it}}{t} dt - \frac{i\pi}{2}$$

and

$$f_2(s) = f_1\left(\frac{2s+i}{1-is}\right) = \log s.$$

(See [4; vol. 1, p. 251, ex. 12].) But as a positive result we have

Theorem 1. Given a function $f_1(s)$ for $\operatorname{Re} s > 0$ with the representation

$$f_1(s) = \int_0^{\infty} e^{-st} F(t) dt$$

such that the following integrals exist:

$$(i) \quad \Phi(t) = \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) \sqrt{\tau} J_1\left(\frac{2}{\gamma} \sqrt{t\tau}\right) d\tau \quad \text{for } t > 0$$

$$(ii) \quad \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) d\tau$$

where $J_1(z)$ denotes the Besselfunction

$$J_1(z) = \sum_{v=0}^{\infty} \frac{(-1)^v \left(\frac{z}{2}\right)^{2v+1}}{v!(v+1)!}.$$

Then

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st} e^{-\frac{i\delta t}{\gamma}} t^{-\frac{1}{2}} \Phi(t) dt + \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) d\tau.$$

The analogue to Theorem B is

Theorem 2. Let $\alpha, \beta, \gamma, \delta$ satisfy (1) and (2). Then there exists a function $f_1(s)$ having for $\operatorname{Re} s > 0$ the form

$$f_1(s) = \int_0^\infty e^{-st} F_1(t) dt + K_3$$

with

$$\int_0^\infty |F_1(t)| dt < \infty,$$

but such that

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st} F_2(t) dt + K_4$$

with

$$\int_0^\infty |F_2(t)| dt = \infty.$$

Finally we obtain an analogue of the second part of theorem A.

Theorem 3. Suppose $k \geq 0$ and let $\alpha, \beta, \gamma, \delta$ satisfy (1) and (2). Assume for $\operatorname{Re} s > 0$ the existence of

$$f_{1,k}(s) = \lim_{x \rightarrow \infty} \frac{1}{x^k} \int_0^x (x-t)^k e^{-st} dF_1(t) = \mathcal{L}^k(F_1)^*$$

$$f_{2,k}(s) = f_{1,k}(\varphi(s)) = \mathcal{L}^k(F_2).$$

*) For this notation see [3, p. 314], [6, p. 4].

a) Then the existence of $f_{1,k}(s)$ at $s=0$ implies

$$\lim_{r \rightarrow 0+} f_{2,k} \left(-\frac{i\beta}{\alpha} + r \right) = \lim_{r \rightarrow 0+} f_{1,k}(r),$$

i.e. if $f_{1,k}(0)$ exists then $f_{2,k}(s)$ is Abel-summable at $s = \varphi^{-1}(0) = -\frac{i\beta}{\alpha}$.

b) If $f_{1,k}(s)$ is bounded in a Stolz-region, i.e. for s with

$$|\arg s| < \psi < \frac{\pi}{2}, \quad |s| < 1,$$

then the existence of $\lim_{r \rightarrow 0+} f_{1,k}(r)$ implies

$$\lim_{r \rightarrow 0+} f_{2,k} \left(-\frac{i\beta}{\alpha} + r \right) = \lim_{r \rightarrow 0+} f_{1,k}(r),$$

i.e. if $f_{1,k}(s)$ is Abel-summable at $s=0$ then the same is true for $f_{2,k}(s)$ at $s = \varphi^{-1}(0) = -\frac{i\beta}{\alpha}$.

PROOF OF THEOREM 1: It is known [4; vol. 1, p. 245, ex. 36] that for $a \geq 0$ and $\operatorname{Re} s > 0$

$$1 - e^{-\frac{a}{s}} = \int_0^\infty e^{-st} \sqrt{\frac{a}{t}} J_1(2\sqrt{at}) dt$$

hence

$$1 - \exp \left(-\frac{\tau}{\gamma} \frac{1}{\gamma s + i\delta} \right) = \frac{1}{\gamma} \int_0^\infty e^{-st} e^{-\frac{i\delta t}{\gamma}} \sqrt{\frac{\tau}{t}} J_1 \left(\frac{2}{\gamma} \sqrt{\tau t} \right) dt.$$

Theorem 1 is proved if in the following repeated integrals the change in the order of integration may be justified. Since (i) and (ii) we have

$$\begin{aligned} \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) d\tau - \int_0^\infty e^{-\varphi(s)\tau} F(\tau) d\tau &= \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) \left\{ 1 - \exp \left(-\frac{\tau}{\gamma} \frac{1}{\gamma s + i\delta} \right) \right\} d\tau = \\ &= \frac{1}{\gamma} \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) \int_0^\infty e^{-ts} e^{-\frac{i\delta t}{\gamma}} \sqrt{\frac{\tau}{t}} J_1 \left(\frac{2}{\gamma} \sqrt{\tau t} \right) dt d\tau = \\ &= \frac{1}{\gamma} \int_0^\infty e^{-st} e^{-\frac{i\delta t}{\gamma}} t^{-\frac{1}{2}} \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau) \sqrt{\tau} J_1 \left(\frac{2}{\gamma} \sqrt{\tau t} \right) d\tau dt. \end{aligned}$$

The proof of the change in the order of integration is similar to the proof of

$$\int_0^\infty e^{-st} \sqrt{t} \int_0^\infty J_1(2\sqrt{\tau t}) G(\tau) d\tau dt = \frac{1}{s^2} \int_0^\infty e^{-\frac{t}{s}} \sqrt{t} G(t) dt.$$

For this see [3, p. 133] or [8].

PROOF OF THEOREM 2: Theorem B was proved by connecting the coefficients a_n and $b_n(\zeta)$ by a non-absolute-regular matrix method. We cannot generalize this proof since in general $f_1(\varphi(s))$ is no Laplace-integral.

We prove theorem 2 by giving an example of functions satisfying the conditions of theorem 1.

Let $\alpha, \beta, \gamma, \delta$ satisfy (1) and (2) and define $F_1(t)$ by

$$F_1(t) = \begin{cases} e^{\frac{i\alpha t}{\gamma}} \frac{1}{t\sqrt{t-1}} & t > 1 \\ 0 & t \leq 1 \end{cases}.$$

It is known that [4; vol. 2, p. 18, ex. 5]

$$\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F_1(\tau) \sqrt{\tau} J_1\left(\frac{2}{\gamma} \sqrt{\tau t}\right) d\tau = \gamma \cdot t^{-\frac{1}{2}} \cdot \sin \frac{2}{\gamma} \sqrt{t}$$

and [4; vol. 1, p. 136, ex. 26] for $\operatorname{Re} s \geq 0$

$$f_1(s) = \pi \operatorname{Erf} c \left(\sqrt{s - \frac{i\alpha}{\gamma}} \right) = \int_0^\infty e^{-st} F_1(t) dt = \pi - 2\sqrt{\pi} \int_0^{\sqrt{s - \frac{i\alpha}{\gamma}}} e^{-t^2} dt.$$

From Theorem 1 and [4; vol. 1, p. 154, ex. 34] we deduce

$$F_2(t) = -e^{-\frac{i\delta t}{\gamma}} \cdot \gamma \cdot t^{-\frac{1}{2}} \sin \frac{2}{\gamma} \sqrt{t}$$

and for $\operatorname{Re} s \geq 0, s \neq -i\delta/\gamma$

$$\pi - f_2(s) = \pi \operatorname{Erf} c \left(\frac{1}{\gamma} \sqrt{s + \frac{i\delta}{\gamma}} \right) = \int_0^\infty e^{-st} F_2(t) dt = 2\sqrt{\pi} \int_0^{\frac{1}{\gamma} \sqrt{s + \frac{i\delta}{\gamma}}} e^{-t^2} dt.$$

Hence we have

$$\int_0^\infty |F_1(t)| dt = \int_1^\infty \frac{1}{t\sqrt{t-1}} dt < \infty$$

and

$$\int_0^\infty |F_2(t)| dt = \int_0^\infty \frac{\left| \sin \frac{2}{\gamma} \sqrt{t} \right|}{t} dt = \infty.$$

PROOF OF THEOREM 3: Let $r > 0$. We have $f_{2,k} \left(r - \frac{i\beta}{\alpha} \right) = f_{1,k} \left(\frac{\alpha^2 r}{1 - i\alpha\gamma r} \right)$ and for $\operatorname{Re} s > 0$ the functions $f_{1,k}(s), f_{2,k}(s)$ are holomorphic (see [3, p. 330], [7, p. 284]). $\frac{\alpha^2 r}{1 - i\alpha\gamma r}$ approaches 0 inside an angle $< \pi$ with vertex at 0 so that the generalized

Abel-limitation theorem can be applied [3, p. 331], [7, p. 284] in case a). By proving case *b* in a similar way we use theorem I of [2, p. 457].

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