

On locally C^* -algebras

By B. D. MALVIYA (Denton, Texas)

1. Introduction

As is well known the concept of Banach algebras have been generalized to locally m -convex algebras by MICHAEL [5]. He obtained many results on locally m -convex algebras with and without involution. Some results concerning spectrum, topological divisors of zero, and the representations of locally m -convex $*$ -algebra have been obtained in [2], [3], [4]. INOUE [1] has recently introduced the notion of locally C^* -algebras. The main objective of the present paper is to obtain further results on locally C^* -algebras by using some of the techniques developed by INOUE [1]. In particular we consider the representation of a locally C^* -algebra on a Hilbert space. We also prove that under certain conditions a locally C^* -algebra admits a complete set of irreducible representations. An inequality involving the positive functionals on a locally C^* -algebra and spectral radius is also obtained.

2. Preliminaries

A topological algebra is an algebra and a topological linear space such that the ring multiplication is continuous. A locally convex algebra is a topological algebra and a locally convex linear space. A topological algebra A with usual involution $*$ such that $x \rightarrow x^* (x \in A)$ is continuous, is called topological $*$ -algebra. A locally m -convex algebra (respectively $*$ -algebra) A is a Hausdorff topological vector space which is an algebra (respectively a $*$ -algebra for which $x \rightarrow x^*$ is continuous) having a fundamental family (U) of circled convex neighbourhoods of $0 \in A$ such that $UU \subset U$. It is clear that every locally m -convex algebra is a locally convex algebra with jointly continuous ring multiplication. The topology in a locally m -convex algebra can be generated by a collection $\{p_j: j \in J\}$ of semi-norms satisfying $p_j(xy) \leq p_j(x)p_j(y)$ for all $x, y \in A$ and all $j \in J$. By an m -base in a locally m -convex algebra A , we understand the collection $\{U\}$ of subsets of A such that U is convex, symmetric and scalar multiples of $\{U\}$ form a base for the neighbourhood of the origin.

Following [1] we define a locally C^* -algebra as follows.

Definition 1.1. A $*$ -algebra A is called a locally C^* -algebra if there exists a family of semi-norms $\{p_j: j \in J\}$ defined on A such that

1. $\{p_j\}$ defines a complete Hausdorff locally convex topology on A .
2. $p_j(xy) \leq p_j(x)p_j(y)$ for all $x, y \in A$ and all $j \in J$.
3. $p_j(x^*) = p_j(x)$ for all $x \in A$ and all $j \in J$.
4. $p_j(x^*x) = p_j(x)^2$ for all $x \in A$ and all $j \in J$.

It may be mentioned that a closed $*$ -subalgebra of a locally C^* -algebra is also a locally C^* -algebra. Our locally C^* -algebras are always complete. Let A be a locally C^* -algebra. Define $N_j = \{x \in A: p_j(x) = 0\}$. Obviously N_j is a closed ideal in A . Let $A_j = A/N_j$ be given a norm V_j defined by $V_j(x + N_j) = p_j(x)$ for $x \in A$. Then A_j is a normed $*$ -algebra with the norm as defined above; and furthermore $\|x_j^* x_j\| = \|x_j\|^2$, where x is any pre-image of x_j under the natural homomorphism π_j . The completion $\bar{A}_j$ of A_j is a C^* -algebra. We shall follow the terminology and notations used by Inoue [1] as closely as possible. For general terms not defined in this paper a reference is made to RICKART [7].

Let A be an algebra. An element $x \in A$ is said to be left (respectively right) quasi-regular if there exists an element $y \in A$ such that $x + y - yx = 0$ (respectively $x + y - xy = 0$). An element which is both left and right quasi-regular is simply called quasi-regular. An element which is not quasi-regular is called quasi-singular. The spectrum of an element $x \in A$, written $\text{Sp}_A(x)$, is above $\{0\}$ the set $\{\lambda: \lambda^{-1}x \text{ is quasi-singular in } A\}$. The spectral radius of an element $x \in A$, written $v_A(x)$, is defined as $v_A(x) = \max \{|\lambda|: \lambda \in \text{Sp}_A(x)\}$. The spectral radius of an element x of a complete locally m -convex algebra A is given by $v_A(x) = \sup_j \lim_{n \rightarrow \infty} (p_j(x^n))^{1/n}$ [5, Corollary 5.3, p. 22].

3. Spectrum

This section mainly concerns with theorems related to spectrum of an element in a locally C^* -algebra. The set of all complex homomorphisms of a commutative algebra A is denoted by Φ_A . For a commutative locally C^* -algebra A , the map $x \rightarrow \hat{x}$ defined by $\hat{x}(\varphi) = \varphi(x)$, ($x \in A$, $\varphi \in \Phi_A$), is called the Gelfand map. The set of hermitian elements of a $*$ -algebra A is denoted by H_A . With the aid of [1, Theorem 3.1, p. 212 and Proposition 3.1, p. 211], it can be proved that the spectrum of an element x of a complete, commutative locally m -convex algebra A is given by the set $\{\hat{x}(\varphi): \varphi \in \Phi_A\}$. Observe that an element $x \in A$ is quasi-regular if and only if $\hat{x}(\varphi) \neq 1$.

Definition 3.1. A $*$ -algebra is called symmetric if x^*x is quasi-regular for every $x \in A$.

Theorem 3.2. Let A be a locally C^* -algebra with the family $\{p_j\}$ $j \in J$ of semi-norms and B a symmetric sublocally C^* -algebra of A . If $x \in B$ is quasi-regular in A , then x is also quasi-regular in B .

PROOF. Since $h = x^*0x (= x^* + x - x^*x)$ is quasi-regular in A , therefore $1 \notin \text{Sp}_A(h)$. Now by [1, Corollary 2.2, p. 200], $\text{Sp}_A(h)$ is real. Again using [1, Corollary 2.3, p. 200], it follows that $\text{Sp}_A(h) = \text{Sp}_B(h)$. Hence $1 \notin \text{Sp}_B(h)$ and therefore h is quasi-regular in B . Similar statement holds for x^*0x . Hence the theorem is proved.

Theorem 3.3. *Let A be a commutative and symmetric locally C^* -algebra without an identity element and A_1 , the $*$ -algebra obtained by adjunction of an identity element to A . Then A_1 is symmetric.*

PROOF. The fact that A_1 is a locally C^* -algebra follows from [1, Theorem 2.3, p. 201]. For an arbitrary element $y = \xi + x \in A_1$, put $h = \bar{\xi}x^* + \xi x + x^*x$. Then $h \in H_A$. Let C be a maximal $*$ -subalgebra of A containing h . It is easy to see that C is a closed $*$ -subalgebra of A and therefore a symmetric sublocally C^* -algebra of A . Let ϕ be an arbitrary element Φ_{A_1} . Then by [1, Corollary 2.3, p. 200] applied to C , we have $y \hat{*} y(\phi) = |\xi|^2 + \hat{h}(\phi) \neq 1$. Hence the theorem is proved.

4. Representations

As mentioned in the introduction we shall deal with the positive functionals and $*$ -representations on a locally C^* -algebra. We shall use the same notations as employed by Inoue [1, pp. 211—213].

Let A be a locally C^* -algebra with a family $\{p_j\}$, $j \in J$ of semi-norms. Define $U_j = \{x \in A : p_j(x) \leq 1\}$. A^* and $A^*(j)$ will denote respectively the conjugate space of A and all linear functionals on A that are bounded on U_j .

Let A be a $*$ -algebra and f a positive functional, that is $f(x^*x) \geq 0$ for each $x \in A$. Define $(x, y) = f(y^*x)$. The following is well known for any $x, y \in A$.

- (i) $f(y^*x) = \overline{f(x^*y)}$.
- (ii) $|f(x^*y)|^2 \leq f(x^*x)f(y^*y)$.

In presence of an identity element, putting $y=e$ in (i) and (ii), we get

- (iii) $f(x^*) = \overline{f(x)}$.
- (iv) $|f(x)|^2 \leq f(e)f(x^*x)$.

A positive functional f on A is said to be admissible if

$$\sup_{x \in A} \frac{f(x^*a^*ax)}{f(x^*x)} < \infty,$$

for each $x, a \in A$ and $f(x^*x) \neq 0$.

Theorem 4.1. *Let A be a locally C^* -algebra with an identity element e and $f \in A^*(j)$. Then*

$$|f(x)|^2 \leq (f(e))^2 v(x^*x)$$

for each $x \in A$.

PROOF. From the proof of Lemma 4.1.4 in [7, p. 183] it follows that there exists an element $y \in A$ such that $y^2 = e - x$ for $x \in A$, provided $p_j(x^n)^{1/n} < 1$ (or $v(x) < 1$); see [5, Corollary 5.3, p. 22]. It also follows from the lemma cited above that if x

is hermitian, then y is also hermitian and $y^2 = y^*y = e - x$. Let $\varepsilon > 0$ be given. Then $v\left(\frac{x}{v(x) + \varepsilon}\right) < 1$ and we have

$$y^2 = e - \frac{x}{v(x) + \varepsilon}.$$

If x is hermitian, then

$$0 \leq f(y^*y) \leq f(e) - \frac{f(x)}{v(x) + \varepsilon}.$$

Since ε is arbitrary, we get

$$f(x) \leq f(e)v(x).$$

Replacing x by x^*x , we get

$$(1) \quad f(x^*x) \leq f(e)v(x^*x).$$

By [1, Theorem 3.2, p. 213], we have

$$(2) \quad |f(x)|^2 \leq \|f_j\| f(x^*x)$$

for each $x \in A$ and $j \in J(f)$, where $J(f) = \{j \in J : f \text{ bounded on } U_j\}$.

From (1) and (2),

$$|f(x)|^2 \leq f(e)\|f_j\|v(x^*x).$$

Using [1, Corollary 3.2, p. 216], we get

$$|f(x)|^2 \leq (f(e))^2 v(x^*x),$$

which is the desired inequality.

Let f be a positive functional on a^* -algebra A . Define $N_f = \{x \in A : f(x^*x) = 0\}$. Obviously N_f is a left ideal in A . For elements x_f and y_f in the difference space A/N_f ; define $f(y^*x) = (x_f, y_f)$. It is easily verified that A/N_f with this inner product is a pre-Hilbert space. The completion of A/N_f will be denoted by H_f .

In the theorems to follow a reference to Hilbert space means the Hilbert space H_f as constructed above.

Definition 4.2. A representation $a \rightarrow T_a$ of a $*$ -algebra A on a Hilbert space is called a $*$ -representation if $T_{a^*} = T_a^*$, where T_a^* is the adjoint of T_a ; $a \in A$.

Definition 4.3. A representation $a \rightarrow T_a$ of an algebra A on a Hilbert space H is called topologically cyclic if there exists a vector $f \in H$ such that the linear subspace $\{T_a f : a \in A\}$ is dense in H .

Theorem 4.4. Let A be a locally C^* -algebra with an identity element e . Suppose that A has bounded spectrum. Then there exists a $*$ -representation $a \rightarrow T_a$ of A on a Hilbert space which is topologically cyclic with a cyclic vector ζ such that $f(a) = (T_a \zeta, \zeta)$, where $f \in A^*(j)$ and is a positive functional.

PROOF. The construction of Hilbert space is obvious as indicated before. Let us write the inner product as $f(y^*x) = (x_f, y_f)$. Define an operator T_x on A/N_f by

$T_x(y+N_f)=xy+N_f$. The mapping $x \rightarrow T_x$ is obviously a representation. Moreover, $T_x, x \in A$, is bounded, since for $f_0 \in A/N_f, f_0=y+N_f$ we have

$$\begin{aligned}(T_x f_0, T_x f_0) &= (xy+N_f, xy+N_f) = ((xy)_f, (xy)_f) = f((xy)^* xy) \cong \\ &\cong (f(e))^2 v((xy)^* xy)^2,\end{aligned}$$

by Theorem 4.1 above. The extension of T_x to H_f is bounded and unique. The following identity shows that $x \rightarrow T_x$ is a $*$ -representation on A/N_f and hence on H_f . For $x_f, y_f \in A/N_f$, we have

$$(T_a x_f, y_f) = f(y^* a x) = f((a^* y)^* x) = (x_f, T_{a^*} y_f).$$

Now by [1, Theorem 3.2 (2), p. 213], f is admissible; see [7, p. 213]. Hence our theorem follows by an application of [7, Theorem 4.5.4 (iii), p. 215], noting that f is a positive hermitian functional.

We shall remove the identity restriction from Theorem 4.4 above. The details of the proof are the same as in Theorem 4.4, so we only give a skeleton of the proof.

Theorem 4.5. *Let A be a locally C^* -algebra without an identity element and with bounded spectrum. Let $f \in A^*(j)$ be a positive functional. Then there exists a $*$ -representation $x \rightarrow T_x$ of A on a Hilbert space which is topologically cyclic with a cyclic vector ζ_2 such that $f(x) = (T_x \zeta_2, \zeta_2)$.*

PROOF. Let f be a positive functional on A . Let A_1 be the locally C^* -algebra obtained by adjunction of an identity element; see [1, Theorem 2.3, p. 200]. By [1, Theorem 3.2, p. 213], f can be extended uniquely to $\tilde{f}$. Now as in the proof of Theorem 4.4 above, $f(x) = (T_x \zeta, \zeta)$, where $\zeta \in H_f$ is a cyclic vector. Here $x \rightarrow T_x$ is a $*$ -representation corresponding to A_1 . We are now looking for a Hilbert space and a cyclic vector when $\tilde{f}$ is restricted to A . Define $H_1 = \{\zeta \in H : T_x \zeta = \zeta \text{ for all } x \in A\}$. Write $\zeta = \zeta_1 + \zeta_2$, where $\zeta_1 \in H_1$ and $\zeta_2 \in H_1^\perp$ (the orthogonal complement of H_1). Observing the fact that the under the $*$ -representation $x \rightarrow T_x$, H is invariant, we have $f(x) = (T_x \zeta_2, \zeta_2)$. Define $H_0 = \overline{\{T_x \zeta_2 : x \in A\}}$. Obviously H_0 is invariant. By [6, Lemma 4.4.1, p. 206], we have a cyclic vector $\zeta_2 \in H_0$. This proves our assertion completely.

Definition 4.6. The set of representations is said to be complete if for every element $x_0 \neq 0$ of the algebra A , there exists an irreducible representation $x \rightarrow T_x$ such that $T_{x_0} \neq 0$.

We now obtain a theorem corresponding to Theorem 3 of [5, p. 267].

Theorem 4.7. *Let A be a locally C^* -algebra with an m -base. A has a complete set of irreducible representations provided there exist positive functionals f such that $M = N_f$, where M 's are maximal modular ideals in A .*

PROOF. As in Theorem 4.4, $x \rightarrow T_x$ is a $*$ -representation of A on A/N_f . By [4, Theorem 8.4, p. 36], A is semi-simple. Let $0 \neq a \in A$. Let us choose a positive functional f such that $a \notin M$. Define $T_x(y+N_f) = xy+N_f$. Obviously $x \rightarrow T_x$ is irreducible. But the kernel of the homomorphism $x \rightarrow T_x$ is contained in $N_f = M$, therefore $T_a \neq 0$. Hence the theorem is proved.

Lemma 4.8. *Let A be a locally C^* -algebra with an identity element e . Suppose that A has bounded spectrum. Then $\|T_x\|^2 \leq v(x^*x)$ for each $x \in A$, where $x \rightarrow T_x$ is a representation of A .*

PROOF. As in the proof of Theorem 4.4, we have $f(x) = (T_x \zeta, \zeta)$, where $f \in A^*(J)$. Now

$$f(x^*x) = (T_{x^*x} \zeta, \zeta) \leq f(e) v(x^*x),$$

from inequality (1) in the proof of Theorem 4.1. This implies

$$\|T_x \zeta\|^2 \leq \|\zeta\|^2 v(x^*x)$$

or

$$\|T_x\|^2 \leq v(x^*x).$$

Now we state the following theorem about the direct sum of the $*$ -representations. The construction of proof is similar to that in [7, pp. 197—198] and makes use of Lemma 4.8.

Theorem 4.9. *Let A be a locally C^* -algebra with an identity element and bounded spectrum. Then the direct sum of the $*$ -representations is defined**

References

- [1] A. INOUE, Locally C^* -algebras, *Memoirs of the Faculty of Science, Kyushu University*, **35** (1971), 197—235.
- [2] T. HUSAIN and B. D. MALVIYA, On representations of locally m -convex $*$ -algebras. *To appear in Mathematica Japonica* (1972).
- [3] B. D. MALVIYA, Topological divisors of zero in locally m -convex algebras, *to appear*.
- [4] B. D. MALVIYA, On locally m -convex algebras. *To appear in Mathematische Nachrichten* (1972).
- [5] E. A. MICHAEL, Locally multiplicatively convex topological algebras, (1952). Revised 1968.
- [6] M. A. NAIMARK, Normed rings, *Groningen* 1964.
- [7] C. E. RICKART, Banach algebras, *New York*, 1960.

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