

A number-theoretic identity

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1. An arithmetic function $f(n, r)$ is said to be an 'Even function of $n \pmod{r}$ ' ([1]) if $f(n, r) = f((n, r), r)$ for all values of n and $r \geq 1$. ECKFORD COHEN [1] has shown that $f(n, r)$ is even $\pmod{r}$ if and only if it possesses a Fourier Expansion of the form

$$(1.1) \quad f(n, r) = \sum_{d|r} \alpha(d, r) C(n, d)$$

where $C(n, r)$ is *Ramanujan's Trigonometric Sum* defined by

$$C(n, r) = \sum_{\substack{h \pmod{r} \\ (h, r) = 1}} \exp(2\pi i h n / r)$$

and $\alpha(d, r)$ is determined by the formula

$$(1.2) \quad \alpha(d, r) = \frac{1}{r} \sum_{d|r} f(r/\delta, r) C(r/d, \delta)$$

The *Cauchy-Composition* $\pmod{r}$ [2] of two even functions $\pmod{r}$ namely $f(n, r)$ and $g(n, r)$ is defined by

$$(1.3) \quad h(n, r) = \sum_{n \equiv a+b \pmod{r}} f(a, r) g(b, r)$$

the summation extending over $a, b \pmod{r}$ such that $n \equiv a+b \pmod{r}$.

If $f(n, r)$ has the representation (1.1) and

$$(1.4) \quad g(n, r) = \sum_{d|r} \beta(d, r) C(n, d)$$

then it is known ([2]) that the Cauchy product of f and g has the representation

$$(1.5) \quad h(n, r) = r \sum_{d|r} \alpha(d, r) \beta(d, r) C(n, d).$$

Next, let k be a positive integer. If $\varphi_k(r)$ denotes the number of elements of a k -reduced residue system $\pmod{r}$, then $\varphi_k(r)$ is given [3] by

$$(1.6) \quad \varphi_k(r) = \sum_{d|r} \mu(r/d) d^k$$

where $\mu(r)$ is the Möbius Function.

For $k=1$ $\varphi_k(r)$ reduces to Euler's φ -function. It can be easily verified that $\varphi(r)$ divides $\varphi_k(r)$.

We introduce the function $S_k(r)$ defined by

$$(1.8) \quad S_k(r) = \sum_{d|r} \frac{\varphi_k(d)}{\varphi(d)}$$

Clearly, for $k=1$, $S_k(r)=d(r)$, the number of divisors of r .

The aim of this note is to derive an identity (see Theorem in 3.) which generalizes the following interesting relation due to KESAVA MENON [4]:

$$(1.9) \quad \sum_{(a,r)=1} (a-1, r) = \varphi(r)d(r)$$

where the summation on the left is over a reduced residue system (mod r).

2. We need the Fourier representations of two even functions $f(n, r)$ and $q(n, r)$ defined below:

Let

$$(2.1) \quad f(n, r) = (n, r)^k.$$

Then,

$$f(n, r) = \sum_{d|r} \alpha(d, r) C(n, d)$$

where

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta|r} \delta^k C(r/d, r/\delta).$$

Since

$$C(n, r) = \sum_{d|(n, r)} \mu(r/d) d$$

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta|r} \delta^k \sum_{D|(r/d, r/\delta)} \mu(r/D\delta) D,$$

interchanging the order of summation in the right hand side, we have

$$\alpha(d, r) = \frac{1}{r} \sum_{D|r/d} D \sum_{\delta|r/D} \mu(r/D\delta) \delta^k = \frac{1}{r} \sum_{D|r/d} D \varphi_k(r/D),$$

by (1.6). Or

$$(2.2) \quad \alpha(d, r) = \frac{1}{r} \sum_{\delta|r/d} \delta \varphi_k(r/\delta)$$

Now, let

$$(2.3) \quad q(n, r) = \begin{cases} 1, & \text{if } (n, r) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$q(n, r) = \sum_{d|r} \beta(d, r) C(n, d)$$

where

$$\beta(d, r) = \frac{1}{r} \frac{\mu(d) \varphi(r)}{\varphi(d)}$$

3. Theorem. If $(n, r) = 1$,

$$\sum_{(a,r)=1} (|n-a|, r)^k = \varphi(r) s_k(r).$$

PROOF. We note that $\sum_{(a,r)=1} (|n-a|, r)^k$ is the Cauchy product (mod r) of $f(n, r)$ (2.1) and $g(n, r)$ (2.3). Therefore, by (1.5)

$$\sum_{(a,r)=1} (|n-a|, r)^k = \frac{\varphi(r)}{r} \sum_{d|r} \left\{ \sum_{\delta|r/d} \delta \varphi_k \left(\frac{r}{\delta} \right) \right\} \frac{\mu(d)}{\varphi(d)} C(n, d).$$

As $(n, r)=1$, $(n, d)=1$ for every divisor d of r . So, $C(n, d)=\mu(d)$. The sum on the right, then, reduces to

$$\frac{\varphi(r)}{r} \sum_{d|r} \left\{ \sum_{\delta|r/d} \delta \varphi_k \left(\frac{r}{\delta} \right) \right\} \frac{\mu^2(d)}{\varphi(d)} = \frac{\varphi(r)}{r} \sum_{\delta|r} \delta \varphi_k \left(\frac{r}{\delta} \right) \sum_{d|r/\delta} \frac{\mu^2(d)}{\varphi(d)}.$$

As

$$\sum_{d|r} \frac{\mu^2(d)}{\varphi(d)} = \frac{r}{\varphi(r)},$$

further simplification leads to

$$\sum_{(a,r)=1} (|n-a|, r)^k = \varphi(r) \sum_{\delta|r} \frac{\varphi_k \left(\frac{r}{\delta} \right)}{\left(\frac{r}{\delta} \right)} \frac{\left(\frac{r}{\delta} \right)}{\varphi \left(\frac{r}{\delta} \right)} = \varphi(r) \sum_{d|r} \frac{\varphi_k(d)}{\varphi(d)} = \varphi(r) s_k(r), \text{ by (18.)}$$

This completes the proof of the Theorem.

Corollary: (1.9) is the special case of the above theorem for $k=1, n=1$.

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References

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