

Semigroups with idempotent ideals

By G. SZÁSZ (Budapest)

To Professor A. Rapcsák on his 60th birthday

1. By a *semigroup with idempotent ideals* we mean a semigroup in which every ideal is idempotent. The class of commutative semigroups with idempotent ideals was characterized by S. LAJOS ([3], Corollary) as follows: a commutative semigroup is regular if and only if it is a semigroup with idempotent ideals. In case of omitting the condition of commutativity the "only if" part of the theorem still remains valid (see Proposition 1 below).

In Section 2 we get to a characterisation of semigroups with idempotent ideals through two preparatory theorems and we obtain S. Lajos's theorem mentioned above as a corollary. In Section 3 we give some information about the place of this class of semigroups among other classes of semigroups. (For a characterisation of a quite different kind of these semigroups see [2].)

For notions and concepts we refer to [1] with the only exception that the principal ideal generated by the element a will be denoted by (a) .

2. First we give a condition that is necessary and sufficient for a principal ideal to be idempotent.

Theorem 1. *The principal ideal (a) of a semigroup S is idempotent if and only if $a \in SaSaS$.*

PROOF. Assume $a \in SaSaS$. Then

$$a \in (Sa)(SaS) \subseteq (a)(a) = (a)^2,$$

whence $(a) \subseteq (a)^2$. As the converse inclusion is always true, $(a) = (a)^2$.

Assume now $(a) = (a)^2$. Then

$$\begin{aligned} a \in (a) &= (a)^5 = S^1 a S^1 \cdot S^1 a S^1 \cdot S^1 a S^1 \cdot S^1 a S^1 \cdot S^1 a S^1 = \\ &= S^1 a S^1 S^1 \cdot a \cdot S^1 S^1 a S^1 S^1 \cdot a \cdot S^1 S^1 a S^1 \subseteq \\ &\subseteq SaSaS, \end{aligned}$$

as asserted.

Theorem 2. *Let S be a semigroup and I an ideal of S . If $(a) = (a)^2$ for every element a of I , then $I = I^2$, too.*

PROOF. Assume $(a) = (a)^2$ for each $a \in I$. Then $a \in (a)^2 \subseteq I^2$ for each $a \in I$. Hence $I \subseteq I^2$. On the other hand, $I^2 \subseteq I$ for every ideal I .

Corollary (S. Lajos's theorem). *A commutative semigroup is regular if and only if it is a semigroup with idempotent ideals.*

PROOF. If S is a semigroup with idempotent ideals, then every element a of S can be represented in the form

$$a = xayaz = a(xyz)a \quad (x, y, z \in S)$$

by Theorem 1 and by commutativity. Thus, S is regular.

Assume, conversely, that S is regular and let a be an arbitrary element of S . Then there exists an $x \in S$ such that $axa = a$ and

$$a = (ax)a \in (a)(a) = (a)^2.$$

Hence $(a) = (a)^2$ and thus S is a semigroup with idempotent ideals by Theorem 2.

Remark. The statement converse to this theorem does not hold: $I^2 = I$ does not imply $(a)^2 = (a)$ for each element a of an ideal I (consider $I = S$ in Example 2 below).

Now we prove the main theorem of this note.

Theorem 3. *The following assertions are equivalent:*

- (A) *S is a semigroup with idempotent ideals;*
- (B) *S is a semigroup with idempotent principal ideals;*
- (C) *$a \in SaSaS$ for every element a of the semigroup S .*

PROOF. (A) implies (B) trivially. (B) implies (C) by Theorem 1. (C) implies (A) as follows: if (C) is satisfied, then every principal ideal of S is idempotent by Theorem 1 and thus every ideal of S is idempotent, too, by Theorem 2.

3. In the second half of the proof of Corollary to Theorem 2 we have verified the following

Proposition 1. *Every ideal of a regular semigroup is idempotent.*

It can be shown by the same way that every ideal of a left regular or right regular or intra-regular semigroup is idempotent.

It is well-known that a semigroup in which every ideal is prime is a semigroup with idempotent ideals. (This is a simple consequence of Corollary 3.2. in [4].) As the matter of fact, the class of semigroup with idempotent ideals is properly wider than the class of semigroups with prime ideals, even inside the class of commutative semigroups. This assertion can be verified by the following example:

Example 1. Consider the commutative semigroup $S = \{0, a, b\}$ in which every element is idempotent and $ab = 0$. Every ideal of S is idempotent, but the principal ideal (0) is not prime because $ab \in (0)$, $a \notin (0)$ and $b \notin (0)$.

An ideal I of a semigroup S is said to be *reproduced* by S if $SI = IS = I$ ([5], Einleitung). We show that the class of semigroups reproducing their ideals is properly

wider than the class of semigroups with idempotent ideals (see Proposition 2 and Example 2).

Proposition 2. *Every ideal I of a semigroup S with idempotent ideals is reproduced by S .*

PROOF. In fact, $I = I^2 \subseteq SI \subseteq I$ whence $SI = I$. Similarly, $IS = I$.

Example 2. Let $S = \{0, 1, a\}$ be a commutative semigroup in which $a^2 = 0$. Every ideal of S is reproduced by S , but the principal ideal (a) is not idempotent: $(a)^2 = (0) \neq (a)$. We remark that this semigroup is not regular because the equation $axa = a$ cannot be solved.

References

- [1] A. H. CLIFFORD—G. B. PRESTON, The algebraic theory of semigroups I, *Providence*, 1961.
- [2] N. KUROKI, On semigroups whose ideals are all globally idempotent, *Proc. Japan Acad.*, **47** (1971), 143—146.
- [3] S. LAJOS, A remark on regular semigroups, *Proc. Japan Acad.*, **37** (1961), 29—30.
- [4] O. STEINFELD, Negatíván rendezett algebrai struktúrák prímelemeiről (On prime elements of negatively ordered algebras; in Hungarian with German summary), *Magyar Tud. Akad. Math. Fiz. Oszt. Közl.*, **17** (1967), 467—472.
- [5] G. SZÁSZ, Über Halbgruppen, die ihre Ideale reproduzieren, *Acta Sci. Math. Szeged*, **27** (1966), 141—146.

(Received February 7, 1973.)