

Growth of geometric means of an entire function

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1. Introduction

Let $f(z)$ be an entire function of order ϱ . Let ϱ_1 and λ_1 respectively, be the exponent of convergence and lower exponent convergence of the zeros of $f(z)$; so that

$$(1.1) \quad \lim_{r \rightarrow \infty} \frac{\sup \log n(r)}{\inf \log r} = \frac{\varrho_1}{\lambda_1} \quad (0 \leq \lambda_1 \leq \varrho_1 \leq \infty),$$

where $n(r)$ represents the number of zeros of $f(z)$ in the disc

$$D \equiv \{z: |z| \leq r\}.$$

Further, let

$$(1.2) \quad N(r) = \int_{r_0}^r \frac{n(x)}{x} dx.$$

Define the following mean values of $f(z)$:

$$(1.3) \quad G(r) = \exp \left[\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right],$$

$$g(r) = \exp \left[\frac{1}{2\pi r} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x dx d\theta \right].$$

Further, let

$$(1.4) \quad g_k(r) = \exp \left[\frac{k+1}{2\pi r^{k+1}} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x^k d\theta dx \right], \quad 0 < k < \infty.$$

Clearly, if $T(r)$ denotes the Nevanlinna's characteristic function, then

$$G(r) \leq \exp \{T(r)\}$$

$$g_k(r) \leq \exp \left[\frac{k+1}{r^{k+1}} \int_0^r T(x) x^k dx \right],$$

and so the orders of $g_k(r)$ and $G(r)$ do not exceed the orders of $f(z)$.

We remark here that the result of SRIVASTAVA [9]

$$(1.5) \quad \lim_{r \rightarrow \infty} \frac{\sup \log \log G(r)}{\inf \log r} = \frac{\varrho}{\lambda}$$

and of KUMAR [3]

$$(1.6) \quad \lim_{r \rightarrow \infty} \frac{\sup \log \log g_k(r)}{\inf \log r} = \frac{\varrho}{\lambda'}$$

can be negated as $G(r)$ and $g_k(r)$ are solely exprecible in terms of the zeros. We can consider an entire function with large $M(r)$ (maximum modulus) and small number of zeros (for instance $f(z) = \exp(z)$ and $f(z) = \exp(z^p) \cos \sqrt{z}$ where p is any positive integer).*

In view of Jensen's theorem on the zeros of $f(z)$, we note that

$$(1.7) \quad \lim_{r \rightarrow \infty} \frac{\sup \log \log G(r)}{\inf \log r} = \frac{\varrho_1}{\lambda_1} \quad (0 \leq \lambda_1 \leq \varrho_1 \leq \infty),$$

and

$$(1.8) \quad \lim_{r \rightarrow \infty} \frac{\sup \log \log g_k(r)}{\inf \log r} = \frac{\varrho_1}{\lambda_1} \quad (0 \leq \lambda_1 \leq \varrho_1 \leq \infty).$$

In this paper, our aim is to study the growths of $G(r)$ and $g_k(r)$. The results are given in form of theorems.

2.

Theorem 1. *Let $f(z)$ be an entire function of exponent convergence ϱ_1 and lower exponent convergence λ_1 , then*

$$\liminf_{r \rightarrow \infty} \frac{\log G(r)}{n(r)} \leq \frac{1}{\varrho_1} \leq \frac{1}{\lambda_1} \leq \limsup_{r \rightarrow \infty} \frac{\log G(r)}{n(r)}.$$

PROOF. We first prove the latter half involving λ_1 , supposing that $\lambda_1 > 0$. If this is not true, there will be a positive number j such that for all sufficiently large r ,

$$(2.2) \quad \frac{\log G(r)}{n(r)} < \left(\frac{1}{\lambda_1} - j \right).$$

By Jensen's theorem

$$\log G(r) = \int_{r_0}^r \frac{n(x)}{x} + O(1).$$

* This fact has not been pointed out in the reviews of Srivastava's paper [9] MR [28 # 2216] and KUMAR's paper [3] MR [33 # 4280].

Substituting for $\log G(r)$ in (2.2), we have

$$\frac{\int_{r_0}^r \frac{n(x)}{x} dx}{n(r)} < \left(\frac{1}{\lambda_1} - j \right) + O(1) \quad (r \rightarrow \infty)$$

or

$$(2.3) \quad \frac{n(r)}{\int_{r_0}^r \frac{n(x)}{n} dx} > \left(\frac{1}{\lambda_1} - j \right)^{-1} + O(1) \quad (r \rightarrow \infty).$$

Therefore, by the integration of (2.3)

$$\log N(r) > \left(\frac{1}{\lambda_1} - j \right)^{-1} \log r + O(\log r)$$

which in virtue of Lemma 1.4 [1], leads to the contradiction

$$\lambda_1 = \liminf_{r \rightarrow \infty} \frac{\log N(r)}{\log r} \cong \left(\frac{1}{\lambda_1} - j \right)^{-1}.$$

Similarly we prove the rest part of the theorem.

Remark. Our theorem is not only more general than Jain's Theorem 1 [2, Chapter 1] but has a different proof from his as well as shorter and more widely applicable.

3.

Theorem 2. For an entire function $f(z)$ of exponent convergence ϱ_1 and lower exponent convergence λ_1 , we have

$$(3.1) \quad \limsup_{r \rightarrow \infty} \frac{\log G(r)}{n(r) \log r} \leq 1 - \frac{\lambda_1}{\varrho_1}.$$

PROOF. When $\lambda_1 = 0$ or $\varrho_1 = \infty$ (i.e. $\varrho_1^{-1} = 0$), it is obvious from Jensen's theorem. Hence we suppose that $\lambda_1 > 0$, $\varrho_1 < \infty$ and deduce from Jensen's theorem

$$(3.2) \quad \frac{\log G(r)}{n(r) \log r} = O(1) + 1 - J(r)P(r), \quad r \rightarrow \infty$$

where

$$J(r) = \frac{\int_{r_0}^r \log x \, dn(x)}{\int_{r_0}^r \log n(x) \, dn(x)},$$

and

$$P(r) = \frac{\int_{r_0}^r \log n(x) \, dn(x)}{n(r) \log r} = \frac{n(r) \log n(r) - n(r) + \text{a constant}}{n(r) \log r}.$$

Now

$$(3.3) \quad \liminf_{r \rightarrow \infty} J(r) \cong \liminf_{r \rightarrow \infty} \frac{\log r}{\log n(r)} = \frac{1}{\varrho_1}$$

and

$$(3.4) \quad \liminf_{r \rightarrow \infty} P(r) = \liminf_{r \rightarrow \infty} \frac{\log n(r)}{\log r} = \lambda_1,$$

(3.3) and (3.4) in conjunction with (3.2), prove the theorem.

4.

Here we prove more sharper inequalities than those of SRIVASTAVA [10]. In what follows we shall prove the following:

Theorem 3. *For an entire function of finite order $\varrho > 0$, we have*

$$(4.2) \quad \limsup_{r \rightarrow \infty} \frac{\log G(r)}{r^{\varrho} \log r} \cong \limsup_{r \rightarrow \infty} \frac{n(r)}{r^{\varrho}} \cong e\varrho T - \varrho t,$$

and

$$(4.3) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{r^{\varrho} \log r} \cong \liminf_{r \rightarrow \infty} \frac{n(r)}{r^{\varrho}} \cong \lambda_1 T.$$

where T and t are type and lower type ($t \neq 0$) of $f(z)$ respectively.

For the proof of the theorem we require the following lemmas.

Lemma 1. *For any entire function of finite non-zero order ϱ*

$$(4.4) \quad e\varrho T \cong \Delta + \varrho t$$

where

$$\Delta = \limsup_{r \rightarrow \infty} \frac{n(r)}{r^{\varrho}}.$$

PROOF. We suppose $f(0) \neq 0$. By Jensen's theorem

$$\log M(r) \cong O(1) + \int_{r_0}^r \frac{n(x)}{x} dx.$$

Let $\Delta' > 0$ such that $\Delta - \Delta' = \varepsilon > 0$. Suppose $\frac{n(r_1)}{r_1^{\varrho}} > \Delta'$ where $r_1 = r_1(\Delta')$. Then

for all $r > r_1$

$$(4.5) \quad \begin{aligned} \log M(r) &> O(1) + \log M(r_1) + \Delta' r_1^q \int_{r_1}^r \frac{dx}{x} = \\ &= O(1) + \log M(r_1) + \Delta' r_1^q \left[\log \frac{r}{r_1} \right]. \end{aligned}$$

Also, it is possible to choose r_1 such that $\frac{\log M(r_1)}{r_1^q} > t'$ with $t - t' = \varepsilon'$. Therefore, from (4.5) for all $r > r_1$, we obtain

$$(4.6) \quad \frac{\log M(r)}{r^q} > \left(\frac{r_1}{r} \right)^q \left[t' + \Delta' \log \frac{r}{r_1} \right] + O(r^q).$$

Now, by the usual method of calculus we maximise the first term of right hand side of (4.6). We find its maxima which is attained for that value of r which satisfies the relation

$$(4.7) \quad \left(\frac{r}{r_1} \right) = \exp \left[\frac{\Delta' - qt'}{q\Delta'} \right]$$

and that maximum value is

$$\left(\frac{\Delta'}{q} \right) \exp \left[\frac{qt' - \Delta'}{\Delta'} \right].$$

Therefore from (4.6), we get

$$(4.8) \quad \frac{\log M(r)}{r^q} > \left(\frac{\Delta'}{q} \right) \exp \left[\frac{qt' - \Delta'}{\Delta'} \right] + O(r^q)$$

for r satisfying (4.7). We see that

$$(4.9) \quad qT \cong \Delta' \exp \left[\left(\frac{qt'}{\Delta'} \right) - 1 \right].$$

Now, since Δ' can be fixed arbitrary close to Δ and t' arbitrary close to t , we immediately deduce from (4.9) the following result

$$qT \cong \Delta \exp \left[\left(\frac{qt}{\Delta} \right) - 1 \right].$$

Since for every real x , $e^x \cong 1 + x$, we finally get

$$eqT \cong \Delta + qt$$

or

$$\limsup_{r \rightarrow \infty} \frac{n(r)}{r^q} \cong eqT - qt.$$

Lemma 2. [5].

$$(4.10) \quad \liminf_{r \rightarrow \infty} \frac{n(r)}{r^q} \cong qt.$$

Lemma 3.

$$(4.11) \quad \liminf_{r \rightarrow \infty} \frac{n(r)}{r^\varrho} \leq \lambda_1 T.$$

PROOF. It is known that

$$(4.12) \quad \liminf_{r \rightarrow \infty} \frac{n(r)}{\log M(r)} \leq \lambda_1. \quad ([6])$$

Let

$$\frac{(nr)}{r^\varrho} = \frac{n(r)}{\log M(r)} \frac{\log M(r)}{r^\varrho}$$

It is well known that if $\varphi(x)$ and $\Phi(x)$ are two non-negative functions then

$$\liminf \{\varphi(x) \cdot \Phi(x)\} \leq \liminf \varphi(x) \limsup \Phi(x).$$

Here $\frac{n(r)}{\log M(r)}$ and $\frac{\log M(r)}{r^\varrho}$ are non-negative, so we have

$$\liminf_{r \rightarrow \infty} \frac{n(r)}{r^\varrho} \leq \liminf_{r \rightarrow \infty} \frac{n(r)}{\log M(r)} \limsup_{r \rightarrow \infty} \frac{\log M(r)}{r^\varrho}$$

which along with (4.12) and definition of type give us

$$\liminf_{r \rightarrow \infty} \frac{n(r)}{r^\varrho} \leq \lambda_1 T.$$

PROOF OF THEOREM 3. From (1.3), we have

$$\log G(r) = O(1) + \int_{r_0}^r \frac{n(x)}{x} dx \leq n(r)[\log r - \log r_0] + O(1)$$

or

$$\frac{\log G(r)}{r^\varrho \log r} \leq \frac{n(r)}{r^\varrho} + O(r^\varrho).$$

Proceeding to limits and making use of Lemma 1, Lemma 2 and Lemma 3, we get the required inequalities.

5.

In spite of the fact that the functions $\log G(r)$ and $\log g_k(r)$ have the same order and same lower order, it is to be noted that for an entire function $f(z)$ of exponent convergence ϱ_1 ($0 < \varrho_1 < \infty$) the asymptotic relation

$$\log G(r) \sim \log g_k(r) \quad \text{as } r \rightarrow \infty$$

need not be true always, as in the case of ordinary means [4]. Consider for instance an entire function

$$f(z) = \prod_{n=1}^{\infty} [1 + z/n^2]$$

for which

$$\log G(r) \sim r^{\frac{1}{2}}$$

and

$$\log g_k(r) \sim \frac{2(k+1)r^{\frac{1}{2}}}{(2k+3)}.$$

We give below a theorem which gives us information as to how the function $\log G(r)$ and $\log g_k(r)$ grow relative to each other as $r \rightarrow \infty$.

Theorem 4. *Let $f(z)$ be an entire function of finite lower exponent convergence λ_1 , then*

$$(5.1) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{\log g_k(r)} \equiv \left(\frac{\lambda_1}{k+1} \right) \left[\left(1 + \frac{k+1}{\lambda_1} \right)^{\left(1 + \frac{\lambda_1}{k+1} \right)} \right], \quad \lambda_1 > 0$$

$$\equiv 1, \quad \lambda_1 = 0,$$

PROOF. Since

$$\lim_{r \rightarrow \infty} \frac{\sup \log \log G(r)}{\inf \log r} = \frac{\varrho_1}{\lambda_1}, \quad (0 \leq \lambda_1 \leq \varrho_1 \leq \infty).$$

Then, following SHAH [7] there exists a lower proximate order $\lambda_1(r)$ ($0 \leq \lambda_1 < \infty$) relative to $\log G(r)$, satisfying the following conditions:

- (i) $\lambda_1(r)$ is a non-negative continuous function of r for $r \geq r_0 > 0$.
- (ii) $\lambda_1(r)$ is differentiable for all $r > r_0$ except at isolated points at which $\lambda_1'(r-0)$ and $\lambda_1'(r+0)$ exist.
- (iii) $\lim_{r \rightarrow \infty} r \lambda_1'(r) \log r = 0$
- (iv) $\lim_{r \rightarrow \infty} \lambda_1(r) = \lambda_1$
- (v) $r^{\lambda_1(r)} \leq \log G(r)$ and $\liminf_{r \rightarrow \infty} \frac{\log G(r)}{r^{\lambda_1(r)}} = 1$.

From (i)–(iv) deduce that

$$(5.2) \quad \lim_{r \rightarrow \infty} \frac{J(hr)}{J(r)} = h^{\lambda_1}, \quad h > 1 \quad \text{where} \quad J(r) = r^{\lambda_1(r)} \quad ([8]).$$

From (1.4), we have

$$(5.3) \quad \log g_k(r) = \frac{k+1}{r^{k+1}} \int_0^r \log G(x) x^k dx \leq \log G(r).$$

Further

$$(5.4) \quad \log g_k(R) = \frac{k+1}{R^{k+1}} \int_0^R x^k \log G(x) dx \geq \frac{k+1}{R^{k+1}} \int_r^R x^k \log G(x) dx \geq$$

$$\geq \frac{R^{k+1} - r^{k+1}}{R^{k+1}} \log G(r).$$

Let $R = xr$, $x > 1$. Then

$$\log G(r) \cong \frac{x^{k+1}}{x^{k+1}-1} \log g_k(xr)$$

and

$$1 = \liminf_{r \rightarrow \infty} \frac{\log G(r)}{J(r)} \cong \frac{x^{k+1}}{x^{k+1}-1} \liminf_{r \rightarrow \infty} \frac{\log g_k(xr)}{J(r)}.$$

From which it follows that

$$(5.5) \quad \liminf_{r \rightarrow \infty} \frac{\log g_k(xr)}{J(r)} \cong \frac{x^{k+1}-1}{x^{k+1}}.$$

Put

$$\frac{\log g_k(xr)}{J(r)} = \frac{\log g_k(xr)}{J(xr)} \frac{J(xr)}{J(r)}.$$

Here

$$\frac{\log g_k(xr)}{J(xr)} \quad \text{and} \quad \frac{J(xr)}{J(r)}$$

are non-negative and so that

$$\liminf_{r \rightarrow \infty} \frac{\log g_k(xr)}{J(r)} \cong \liminf_{r \rightarrow \infty} \frac{\log g_k(r)}{J(r)} x^{\lambda_1}$$

by (5.2). This inequality with (5.5) gives us

$$\liminf_{r \rightarrow \infty} \frac{\log g_k(r)}{J(r)} \cong \frac{x^{k+1}-1}{x^{k+\lambda_1+1}}.$$

Using this inequality and from the equality

$$\frac{\log G(r)}{\log g_k(r)} = \frac{\log G(r)}{J(r)} \frac{J(r)}{\log g_k(r)},$$

we get

$$(5.6) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{\log g_k(r)} \cong \liminf_{r \rightarrow \infty} \frac{\log G(r)}{J(r)} \limsup_{r \rightarrow \infty} \frac{J(r)}{\log g_k(r)} \cong \frac{x^{(k+\lambda_1+1)}}{x^{k+1}-1}.$$

Now, by the usual method of Calculus we minimize the right hand side of (5.6). We find that its minima is attained for that value of x which satisfies the relation

$$x = [(k+\lambda_1+1)/\lambda_1]^{1/(k+1)}, \quad \lambda_1 > 0.$$

Substituting this value of x in (5.6), we get

$$\liminf_{r \rightarrow \infty} \frac{\log G(r)}{\log g_k(r)} \cong \frac{\lambda_1}{k+1} \left[1 + \frac{k+1}{\lambda_1} \right]^{[1+\lambda_1/(k+1)]}.$$

In the case when $\lambda_1=0$, the minimum value of the right hand side of (5.6) is one as $x \rightarrow \infty$. This completes the proof of the theorem.

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