Some properties of T-matrices over non-archimedian fields

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1. Let $A=(a_{np})$, $n, p=1, 2, \ldots$ be a matrix defined over a field K provided with non-trivial non-archmedian valuation. The field K is supposed to be complete under the metric of valuation. In this note, we shall deduce from a theorem of MONNA [2], the conditions for a matrix A to be regular and study some properties of regular matrices known as T-matrices over K. We note that the classical Steinhaus Theorem dealing with T-matrices [1] is not true in general in the non-archidedian case in § 2. However we recover the Steinhaus theorem for a restricted class of matrices over K in § 3. Suitably defining the absolute equivalence of T-matrices over K, we shall find out anecessary and sufficient condition for a matrix T to be absolutely equivalent for all bounded sequences over K in § 4.

2. From a theorem of MONNA [2], we deduce as in the classical case, the following theorem.

Theorem 1. *A matrix $A=(a_{np})$ is a T-matrix over K called a $T(K)$ matrix if and only if*

(2.1)
$$\operatorname*{Sup}_{n,p} |a_{np}| \leqq M$$

where M is a constant,

(2.2)
$$\lim_{n\to\infty} a_{np} = 0$$

for every fixed p,

(2.3)
$$\sum_{p=1}^{\infty} a_{np} = A_n \to 1 \quad as \quad n \to \infty.$$

Let $y_n = \sum_{p=1}^{\infty} a_{np} x_p$, $n=1, 2, 3, \ldots$. The sequence (y_n) is called the A-transform of (x_p). If the sequence (y_n) is convergent, its limit is called the A-limit. The classical Steinhaus Theorem states that given a T-matrix, there is always a bounded sequence which has no A-limit.

If π is a prime number, we shall construct $T(K)$ matrices over the π-adic field K which are rational number fields completed under π-adic valuation.

Example 1. Let

$$a_{np} = \begin{cases} 1/2 & \text{for} \quad p = n, n+1 \\ 0 & \text{otherwise} \end{cases}$$

This matrix is evidently a $T(K)$ matrix. Consider a bounded sequence $(1, 0, 1, 0, 1, \ldots)$ where 1 is the multiplicative identity of the π-adic field K. A-transforms of this bounded sequence gives rise to a sequence $y_n = 1/2$, $n = 1, 2, \ldots$ which is convergent sequence having the limit $1/2$. Here the A-limit of a bounded sequence exists.

Example 2. Let

$$a_{np} = \begin{cases} \pi^n & \text{for} \quad n > p \\ 1-(1-n)\pi^n & \text{for} \quad n = p \\ 0 & \text{othervise.} \end{cases}$$

One can easily verify that $A = (a_{np})$ defined above is a $T(K)$ matrix over the π-adic field. The A-transforms of this sequence $(1, 0, 1, 0, \ldots)$ gives rise to the sequence y_n defined by

$$y_n = \begin{cases} 1-(n-1)/2\pi^n & \text{if} \quad n \text{ is odd.} \\ n/2\pi^n & \text{if} \quad n \text{ is even.} \end{cases}$$

(y_n) is not convergent sequence. Thus the A-transform of a bounded sequence is not convergent. These two examples establish that the classical Steinhaus Theorem is not true in general in the non-archimedian case.

3. We shall prove the Steinhaus Theorem for a restricted class of regular matrices over K.

Theorem 2. *Let $A = (a_{np})$ be a regular matrix over K with the following restriction on* (2.1)

$$\sup_{n,p} |a_{np}| \leqq \lambda \quad \textit{where} \quad \lambda \geqq 1. \tag{3.1}$$

Then there is a bounded sequence which has no A-limit.

Proof. Let (Z_p) be a sequence defined in K such that $0 \leqq |Z_p| \leqq 1$. By (2.3) choose a n_1 such that the following is satisfied.

$$\left| \sum_{p=1}^{\infty} a_{n_1 p} \right| = A_{n_1} > \mu \quad \text{where} \quad \mu < 1. \tag{3.2}$$

Since $\sum_{p=1}^{\infty} a_{np}$ converges for each fixed n, we have

$$a_{np} \to 0 \quad \text{as} \quad p \to \infty. \tag{3.3}$$

Hence given $\varepsilon > 0$, we can find a p_1 such that $|a_{np}| < \varepsilon$ for $p \geqq p_1$ for every fixed n. Using this we have from the above,

$$\sup_{p_1+1 \leqq p < \infty} |a_{n_1 p}| < \varepsilon. \tag{3.4}$$

Let us choose $z_p=1$ for $1\leqq p\leqq p_1$

$$Z'_{n_1}=\sum_{p=1}^{\infty}a_{n_1 p}-\sum_{p=p_1+1}^{\infty}(1-z_p)a_{n_1 p}.$$

Therefore

$$\sum_{p=1}^{\infty}a_{n_1 p}=Z'_{n_1}+\sum_{p=p_1+1}^{\infty}(1-z_p)a_{n_1 p}.$$

Hence we have from the above

$$\left|\sum_{p=1}^{\infty}a_{n_1 p}\right|\leqq \operatorname{Max}\left\{|Z'_{n_1}|,\ \left|\sum_{p=p_1+1}^{\infty}(1-z_p)a_{n_1 p}\right|\right\} \tag{3.5}$$

But

$$|1-Z_p|=1,\quad \left|\sum_{p=p_1+1}^{\infty}(1-Z_p)a_{n_1 p}\right|\leqq \operatorname*{Sup}_{p_1+1\leqq p<\infty}|a_{n_1 p}|<\varepsilon. \tag{3.6}$$

Substituting (3.6) in (3.5) and using (3.2) and (3.4), we get

$$\mu\leqq \operatorname{Max}\{|Z'_n|,\varepsilon\}\quad \text{wich implies}\quad |Z'_{n_1}|>\mu.$$

By using (3.3), choose a $p_2>p_1$ such that

$$\operatorname*{Sup}_{p_2+1<p<\infty}|a_{n_2 p}|<\varepsilon\quad \text{for arbitrary}\quad \varepsilon>0. \tag{3.7}$$

Let us choose $z_p=0$ for $p_1<p\leqq p_2$

$$z'_{n_2}=\sum_{p=1}^{p_1}a_{n_2 p}z_p+\sum_{p=p_1+1}^{p_2}a_{n_2 p}z_p+\sum_{p=p_2}^{\infty}a_{n_2 p}z_p.$$

Therefore

$$|Z'_{n_2}|\leqq \operatorname{Max}\left\{\left|\sum_{p=1}^{p_1}a_{n_2 p}Z_p\right|,\ \left|\sum_{p_2+1}^{\infty}a_{n_2 p}Z_p\right|\right\}. \tag{3.8}$$

By using (3.1) we have

$$\left|\sum_{p=1}^{p_1}a_{n_2 p}Z_p\right|\leqq \operatorname*{Sup}_{1\leqq p\leqq p_1}|a_{n_2 p}Z_p|<\operatorname*{Sup}_{n,p}|a_{np}|<\lambda. \tag{3.9}$$

By using (3.7) we have

$$\left|\sum_{p_2+1}^{\infty}a_{n_2 p}Z_p\right|\leqq \operatorname*{Sup}_{p_2+1\leqq p<\infty}|a_{n_2 p}||Z_p|<\varepsilon. \tag{3.10}$$

Making use of (3.9) and (3.10) in (3.8) we get

$$|Z'_{n_2}|\leqq \operatorname{Max}\{\lambda,\varepsilon\}\quad \text{so that}\quad |Z'_{n_2}|\leqq\lambda.$$

By (2.3) choose $n_3 > n_2$ such that

$$\left|\sum_{p=1}^{\infty} a_{n_3 p}\right| > \mu \tag{3.11}$$

and also by (2.2)

$$\sup_{1 \leqq p \leqq p_2} |a_{n_3 p}| < \varepsilon. \tag{3.12}$$

By (3.3) choose a $p_3 > p_2$ such that

$$\sup_{p_3+1 \leqq p < \infty} |a_{n_3 p}| < \varepsilon. \tag{3.13}$$

Now we have for Z'_{n_3}

$$Z'_{n_3} = \sum_{p=1}^{\infty} a_{n_3 p} - \sum_{p_1+1}^{p_2} a_{n_3 p} - \sum_{p=p_3+1}^{\infty} (1-Z_p) a_{n_3 p}$$

$$\sum_{p=1}^{\infty} a_{n_3 p} = Z'_{n_3} + \sum_{p_1+1}^{\infty} a_{n_3 p} + \sum_{p=p_3+1}^{\infty} (1-Z_p) a_{n_3 p}$$

$$\left|\sum_{p=1}^{\infty} a_{n_3 p}\right| \leqq \operatorname{Max}\left\{|Z'_{n_3}|, \left|\sum_{p_1+1}^{p_2} a_{n_3 p}\right|, \left|\sum_{p_3+1}^{\infty} (1-Z_p) a_{n_3 p}\right|\right\}. \tag{3.14}$$

Using (3.11), (3.12) and (3.13) in (3.14) we get,

$$\mu \leqq \operatorname{Max}\{|Z'_{n_3}|, \varepsilon, \varepsilon\} \quad \text{from which we have} \quad |Z'_{n_3}| > \mu.$$

By our assumption, λ can never be equal to μ so that (Z'_n) is not convergent. Hence if A is a $T(K)$ matrix satisfying (3.1), we can choose (Z_k) so that all its elements are 0 or 1 and (Z'_n) does not tend to a limit. This completes the proof of the theorem.

4. Two matrices A and B defined over K are said to be absolutely equivalent for a class of sequences (Z_p), whenever $(Z'_p - Z''_p) \to 0$ as $n \to \infty$ where Z'_p and Z''_p are the A and B transforms of the sequence (Z_p).

Theorem 3. *A necessary and sufficient condition that T matrices A and B defined over K are absolutely equivalent for all bounded sequences (Z_n) is that $S_n = \sup_{1 \leqq p < \infty} |C_{np}| \to 0$ as $n \to \infty$, where $C_{np} = a_{np} - b_{np}$.*

Proof. The condition is sufficient. Now

$$Z'_n - Z''_n = \sum_{p=1}^{\infty} (a_{np} - c_{np}) Z_p = \sum_{p=1}^{\infty} C_{np} Z_p.$$

Therefore $\left|\sum_{p=1}^{\infty} C_{np} Z_p\right| \leqq M \sup_{1 \leqq p < \infty} |C_{np}|$, since $|Z_p| \leqq M$ for all p. Hence if $\sup_{1 \leqq p < \infty} |C_{np}| \to 0$ as $n \to \infty$, then $Z'_n - Z''_n \to 0$ as $n \to \infty$. Therefore A and B are absolutely equivalent for all bounded sequences.

Conversely let us assume that $y_n = Z'_n - Z''_n \to 0$ as $n \to \infty$. That is $\sum_{p=1}^{\infty} C_{np} z_p \to 0$ as $n \to \infty$ for every bounded sequence (z_p). Then we shall prove that $S_n \to 0$ as $n \to \infty$.

Consider the bounded sequence $(1, 1, 1, \ldots)$ where 1 is the identity in K. Then

$$Z'_n - Z''_n = \sum_{p=1}^{\infty} C_{np} Z_p = \sum_{p=1}^{\infty} C_{np}.$$

Since Z_n is defined for every bounded sequence, this is defined for $(1, 1, 1, \ldots)$ also. $S_0 \sum_{p=1}^{\infty} C_{np}$ is well defined in the field K. This implies that

$$C_{np} \to 0 \quad \text{as} \quad p \to \infty \quad \text{for every fixed } n. \tag{4.1}$$

Suppose $S_n = \sup_{1 \leqq p < \infty} C_{np}$ does not tend to zero as $n \to \infty$, it will tend to ∞ through a subsequence of values of n. Then for some $\varepsilon > 0$, there exists a subsequence of values of n such that

$$\sup_{1 \leqq p < \infty} |C_{np}| > \varepsilon. \tag{4.2}$$

Using (4.1) we can find a p_{n1} such that

$$\sup_{p_{n_1}+1 \leqq p < \infty} |C_{n_1 p}| < \frac{\varepsilon\lambda}{2} \tag{4.3}$$

where $\lambda < 1$ corresponds to some element $Z \in K$ for which $|Z| = \lambda$. Such an $Z \in K$ exists because the valuation is non-trivial.

Comparing (4.2) and (4.3) we get $\sup_{1 \leqq p \leqq p_{n_1}} |C_{n_1 p}| > \varepsilon$. Therefore there is a p_1 in this range such that

$$|C_{n_1 p_1}| > \varepsilon. \tag{4.4}$$

Now we shall construct a sequence (Z_p) with the condition that $|Z_p| \leqq 1$ and $y_n = Z'_n - Z''_n$ does not tend to zero. Let

$$Z_p = \begin{cases} Z & \text{where} \quad |Z| = \lambda < 1 \quad \text{when} \quad p = p_1 \\ 0 & \text{for all } p \text{ in} \quad 1 \leqq p \leqq p_{n_1} \quad \text{and} \quad p \neq p_1. \end{cases} \tag{4.5}$$

Now

$$y_{n_1} = \sum_{p=1}^{p_{n_1}} C_{n_1 p} Z_p + \sum_{p_{n_1}+1}^{\infty} C_{n_1 p} Z_p. \tag{4.6}$$

But

$$\left| \sum_{p_{n_1}+1}^{\infty} C_{n_1 p} Z_p \right| \leqq \sup_{p_{n_1}+1 \leqq p < \infty} |C_{n_1 p}| < \frac{\varepsilon\lambda}{2},$$

$$\left| \sum_{p=1}^{p_{n_1}} C_{n_1 p} Z_p \right| = |C_{n_1 p_1}| |Z_{p_1}| = |C_{n_1 p_1}| |Z|.$$

Therefore we get from (4.6)

$$|C_{n_1 p_1}| |Z_{p_1}| \leqq \operatorname{Max} \left\{ |y_{n_1}|, \left| \sum_{p=p_{n_1}+1}^{\infty} C_{n_1 p} Z_p \right| \right\}.$$

Therefore $\varepsilon\lambda < \text{Max}\,(|y_{n_1}|, \varepsilon\lambda/2)$ by using (4.3), (4.4) and (4.5). Hence we get

$$|y_{n_1}| > \varepsilon\lambda. \tag{4.7}$$

By (4.1) we have $a_{np} \to 0$ as $n \to \infty$ for each fixed p. Now choose $n_2 > n_1$ such that

$$\sup_{1 \leqq p < \infty} C_{n_2 p} > \varepsilon \tag{4.8}$$

and

$$\sup_{1 \leqq p \leqq p_{n_1}} C_{n_2 p} < \frac{\varepsilon\lambda}{2}. \tag{4.9}$$

This is possible if n_2 is large enough, such that $n_2 > \text{Max}\,(n_p)$ where $1 \leqq p \leqq p_{n_1}$ defined in (4.9). Then there exists by (4.1) a $p_{n_2} > p_{n_1}$ such that

$$\sup_{p_{n_2}+1 \leqq p < \infty} |C_{n_2 p}| < \frac{\varepsilon\lambda}{2}. \tag{4.10}$$

Therefore from (4.8) and (4.10),

$$\sup_{1 \leqq p \leqq p_{n_2}} |C_{n_2 p}| > \varepsilon. \tag{4.11}$$

So we can find a p_2 in $1 \leqq p \leqq p_2$ such that

$$|C_{n_2 p_2}| > \varepsilon \tag{4.12}$$

and p_2 chosen in (4.12) exceeds p_{n_1} by (4.9) and (4.8). Let us define

$$Z_p = \begin{cases} Z & \text{when} \quad p = p_2 \quad \text{and} \quad |Z| = \lambda < 1 \\ 0 & \text{for all } p \quad \text{in} \quad p_{n_1+1} \leqq p \leqq p_{n_2} \quad \text{and} \quad p \neq p_2, \end{cases} \tag{4.13}$$

$$y_{n_2} = \sum_{p=1}^{p_{n_1}} C_{n_2 p} Z_p + \sum_{p_{n_1}+1}^{p_2} C_{n_2 p} Z_p + \sum_{p_{n_2}+1}^{\infty} C_{n_2 p} Z_p$$

$$\left| \sum_{p_{n_2}+1}^{p_{n_2}} C_{n_2 p} Z_p \right| = \left| y_{n_2} - \sum_{p=1}^{p_{n_1}} C_{n_2 p} Z_p - \sum_{p_{n_2}+1}^{\infty} C_{n_2 p} Z_p \right|,$$

$$|C_{n_2 p_2}|\,|Z_p| \leqq \text{Max} \left\{ |y_{n_2}|, \left| \sum_{p=1}^{p_{n_1}} C_{n_2 p} Z_p \right| \left| \sum_{p_{n_2}+1}^{\infty} C_{n_2 p} Z_p \right| \right\} \tag{4.14}$$

by using (4.13) in the left hand side of (4.14). By (4.10) we have

$$\left| \sum_{p_{n_2}+1}^{\infty} C_{n_2 p} Z_p \right| \leqq \sup_{p_{n_2}+1 \leqq p < \infty} |C_{n_2 p}| < \frac{\varepsilon\lambda}{2}. \tag{4.15}$$

From (4.9) we get

$$\left| \sum_{p=1}^{p_{n_1}} C_{n_2 p} Z_p \right| \leqq \sup_{1 \leqq p \leqq p_{n_1}} |C_{n_2 p}| < \frac{\varepsilon\lambda}{2}. \tag{4.16}$$

Using (4.12), (4.13), (4.15) and (4.16) in (4.14) we get

$$\varepsilon\lambda < \mathrm{Max}\left(|y_{n_2}|, \frac{\varepsilon\lambda}{2}, \frac{\varepsilon\lambda}{2}\right)$$

Therefore from the above $|y_{n_2}| > \varepsilon\lambda$. Proceeding in this manner, we can find y_{n_k} such that $|y_{n_k}| > \varepsilon\lambda$ so that $y_n = Z'_n - Z''_n$ does not tend to zero as $n \to \infty$, through a subsequence of values of n. This shows that S_n does not tend to zero for every bounded sequence (Z_n) and this implies that $y_n = Z'_n - Z''_n$ does not tend to zero which is contrary to our assumption. This contradiction proves the necessity of the condition. Hence the theorem is completely proved.

References

[1] Cooke G. Richard, Infinite matrices and sequence spaces, Dover Publications, *New York* (1955).

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