

A few observations regarding continuous solutions of a system of functional equations

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The problem of the existence and uniqueness of the continuous solutions of the system of functional equations

$$(1) \quad \varphi_i(x) = h_i(x; \varphi_1[f_1(x)], \dots, \varphi_1[f_n(x)]; \dots; \varphi_m[f_1(x)], \dots, \varphi_m[f_n(x)]), \\ i = 1, \dots, m,$$

in which $\varphi_i, i = 1, \dots, m$, are unknown functions, was investigated by J. KORDYLEWSKI in [2] in the case where f_k is the k -th iterate of a function $f, k = 1, \dots, n$, under the hypothesis that the characteristic roots of a suitable matrix are less than one in absolute value. Here, we shall give a simpler condition which guarantees the existence and uniqueness of the continuous solutions of system (1) and we shall show that under suitable assumptions this unique solution fulfils a Lipschitz condition. Moreover, we shall prove a theorem about the continuous dependence on the given functions for continuous solutions of this system. The proofs will be based on J. MATKOWSKI's results given in [3].

1. Assume the following hypotheses:

(i) X is a topological space, whereas Y_i with the metric $\varrho_i, i = 1, \dots, m$, are complete metric spaces;

(ii) $h_i: X \times Y_1^n \times \dots \times Y_m^n \rightarrow Y_i, i = 1, \dots, m$, and $f_k: X \rightarrow X, k = 1, \dots, n$, are continuous functions. Furthermore,

$$\varrho_i(h_i(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_i(x; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) \leq \\ \leq \sum_{j=1}^m \sum_{k=1}^n a_{i,j,k} \varrho_j(y_{j,k}, \bar{y}_{j,k}),$$

for every $x \in X$ and $y_{j,k}, \bar{y}_{j,k} \in Y_j; i, j = 1, \dots, m; k = 1, \dots, n$, where $a_{i,j,k}; i, j = 1, \dots, m; k = 1, \dots, n$, are positive constants.

Write

$$(2) \quad b_{i,j} = \sum_{k=1}^n a_{i,j,k}, \quad i, j = 1, \dots, m,$$

$$(3) \quad b_{\lambda,\mu}^1 = \begin{cases} b_{\lambda,\mu} & \text{for } \lambda \neq \mu \\ 1 - b_{\lambda,\mu} & \text{for } \lambda = \mu \end{cases}; \quad \lambda, \mu = 1, \dots, m,$$

$$(4) \quad b_{\lambda, \mu}^{\kappa+1} = \begin{cases} b_{1,1}^{\kappa} b_{\lambda+1, \mu+1}^{\kappa} + b_{\lambda+1, 1}^{\kappa} b_{1, \mu+1}^{\kappa} & \text{for } \lambda \neq \mu \\ b_{1,1}^{\kappa} b_{\lambda+1, \mu+1}^{\kappa} - b_{\lambda+1, 1}^{\kappa} b_{1, \mu+1}^{\kappa} & \text{for } \lambda = \mu \end{cases};$$

$$\kappa = 1, \dots, m-1; \quad \lambda, \mu = 1, \dots, m-\kappa.$$

Theorem 1. Let hypotheses (i) and (ii) be fulfilled and suppose that X is a compact space. If

$$(5) \quad 0 < b_{\lambda, \lambda}^{\kappa}, \quad \kappa = 1, \dots, m; \quad \lambda = 1, \dots, m+1-\kappa,$$

where the constants $b_{\lambda, \mu}^{\kappa}$, $\kappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\kappa$, are defined by (2)–(4), then system (1) has exactly one continuous solution $\varphi_i: X \rightarrow Y_i$, $i=1, \dots, m$. This solution is given by the formula

$$(6) \quad \varphi_i(x) = \lim_{v \rightarrow \infty} \varphi_{i,v}(x), \quad i = 1, \dots, m, \quad x \in X,$$

where

$$(7) \quad \varphi_{i,v+1}(x) = h_i(x; \varphi_{1,v}[f_1(x)], \dots, \varphi_{1,v}[f_n(x)]; \dots; \varphi_{m,v}[f_1(x)], \dots, \varphi_{m,v}[f_n(x)]),$$

$$i = 1, \dots, m, \quad v = 0, 1, 2, \dots, \quad x \in X,$$

and $\varphi_{i,0}$ is an arbitrary continuous map from X into Y_i , $i=1, \dots, m$.

PROOF. Denote by $\mathcal{C}_i$ the complete metric space of all continuous functions $\varphi: X \rightarrow Y_i$ with the supremum metric d_i , $i=1, \dots, m$, and put

$$(8) \quad T_i(\varphi_1, \dots, \varphi_m)(x) = h_i(x; \varphi_1[f_1(x)], \dots, \varphi_1[f_n(x)]; \dots; \varphi_m[f_1(x)], \dots, \varphi_m[f_n(x)]),$$

$$\varphi_j \in \mathcal{C}_j; \quad i, j = 1, \dots, m, \quad x \in X.$$

By hypothesis (ii) we have that

$$T_i(\mathcal{C}_1 \times \dots \times \mathcal{C}_m) \subset \mathcal{C}_i, \quad i = 1, \dots, m$$

and

$$(9) \quad d_i(T_i(\varphi_1, \dots, \varphi_m), T_i(\bar{\varphi}_1, \dots, \bar{\varphi}_m)) \leq \sum_{j=1}^m b_{i,j} d_j(\varphi_j, \bar{\varphi}_j),$$

$$\varphi_j, \bar{\varphi}_j \in \mathcal{C}_j; \quad i, j = 1, \dots, m,$$

where $b_{i,j}$; $i, j=1, \dots, m$, are defined by (2). Thus we may apply Matkowski's theorem contained in [3] from which we obtain our assertion.

In the next theorem the compactness of X is replaced by the hypothesis

(iii) There exists a sequence $\{G_\tau\}$ of open sets such that $X = \bigcup \{G_\tau: \tau=1, 2, \dots\}$, $G_\tau \subset G_{\tau+1}$ and $\bar{G}_\tau$ is compact, $\tau=1, 2, \dots$. Moreover, $f_k(G_\tau) \subset G_\tau$ for $k=1, \dots, n$ and $\tau=1, 2, \dots$.

Namely, we have

Theorem 2. If hypotheses (i)–(iii) and condition (5) are fulfilled, where the constants $b_{\lambda, \mu}^{\kappa}$, $\kappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\kappa$, are defined by (2)–(4), then system (1) has exactly one continuous solution $\varphi_i: X \rightarrow Y_i$, $i=1, \dots, m$. This solution is given by (6) and (7), where $\varphi_{i,0}$ is an arbitrary continuous map from X into Y_i , $i=1, \dots, m$.

The proof of this theorem is similar to that given in [1], theorem 2.

2. Now, we shall give a theorem regarding a property of the solution just obtained. Suppose that

(iv) (X, ϱ) is a metric space and (Y_i, ϱ_i) , $i=1, \dots, m$, are complete metric spaces;

(v) The functions $h_i: X \times Y_1^n \times \dots \times Y_m^n \rightarrow Y_i$, $i=1, \dots, m$, and $f_k: X \rightarrow X$, $k=1, \dots, n$, fulfil the conditions

$$\begin{aligned} \varrho_i(h_i(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_i(\bar{x}; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) &\leq \\ &\leq a_i \varrho(x, \bar{x}) + \sum_{j=1}^m \sum_{k=1}^n a_{i,j,k} \varrho_j(y_{j,k}, \bar{y}_{j,k}), \end{aligned}$$

and

$$\varrho(f_k(x), f_k(\bar{x})) \leq s_k \varrho(x, \bar{x}),$$

for all $x, \bar{x} \in X$ and $y_{j,k}, \bar{y}_{j,k} \in Y_j$; $i, j=1, \dots, m$; $k=1, \dots, n$, where a_i , $a_{i,j,k}$ and s_k ; $i, j=1, \dots, m$; $k=1, \dots, n$, are positive constants.

Put

$$(10) \quad c_{i,j} = \sum_{k=1}^n a_{i,j,k} s_k, \quad i, j = 1, \dots, m,$$

$$(11) \quad c_{\lambda,\mu}^1 = \begin{cases} c_{\lambda,\mu} & \text{for } \lambda \neq \mu \\ 1 - c_{\lambda,\mu} & \text{for } \lambda = \mu \end{cases}; \quad \lambda, \mu = 1, \dots, m,$$

$$(12) \quad c_{\lambda,\mu}^{\kappa+1} = \begin{cases} c_{1,1}^{\kappa} c_{\lambda+1,\mu+1}^{\kappa} + c_{\lambda+1,1}^{\kappa} c_{1,\mu+1}^{\kappa} & \text{for } \lambda \neq \mu \\ c_{1,1}^{\kappa} c_{\lambda+1,\mu+1}^{\kappa} - c_{\lambda+1,1}^{\kappa} c_{1,\mu+1}^{\kappa} & \text{for } \lambda = \mu \end{cases},$$

$$\kappa = 1, \dots, m-1; \quad \lambda, \mu = 1, \dots, m-\kappa.$$

Theorem 3. Let hypotheses (iv) and (v) be fulfilled and suppose that X is a compact space. If the constants $b_{\lambda,\mu}^{\kappa}$ and $c_{\lambda,\mu}^{\kappa}$, $\kappa=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\kappa$, defined by (2)—(4) and (10)—(12) fulfil conditions (5) and

$$(13) \quad 0 < c_{\lambda,\lambda}^{\kappa}, \quad \kappa = 1, \dots, m; \quad \lambda = 1, \dots, m+1-\kappa,$$

respectively, then system (1) has exactly one continuous solution $\varphi_i: X \rightarrow Y_i$, $i=1, \dots, m$. This solution fulfils a Lipschitz condition.

PROOF. The first part of the above assertion evidently results from theorem 1. We have still to prove that the solution obtained fulfils a Lipschitz condition.

It follows from (11)—(13) that there exist positive numbers $l_1, \dots, l_m$ and a $\vartheta \in (0, 1)$, such that

$$\sum_{j=1}^m c_{i,j} l_j \leq \vartheta l_i, \quad i = 1, \dots, m$$

([3], Lemma). In view of the homogeneity of the above system we may assume that

$$\frac{a_i}{1-\vartheta} \leq l_i, \quad i = 1, \dots, m.$$

This means that the system

$$(14) \quad a_i + \sum_{j=1}^m c_{i,j} l_j \equiv l_i, \quad i = 1, \dots, m,$$

has a positive solution l_i , $i=1, \dots, m$. Let $\mathcal{L}_i$ be the class of all functions $\varphi: X \rightarrow Y_i$ such that

$$(15) \quad \varrho_i(\varphi(x), \varphi(\bar{x})) \equiv l_i \varrho(x, \bar{x}), \quad x, \bar{x} \in X, \quad i = 1, \dots, m,$$

where l_i , $i=1, \dots, m$, are a positive solution of (14). We shall prove that the transformation T_i defined by (8) fulfils

$$(16) \quad T_i(\mathcal{L}_1 \times \dots \times \mathcal{L}_m) \subset \mathcal{L}_i, \quad i = 1, \dots, m.$$

Indeed, suppose that $\varphi_i \in \mathcal{L}_i$, $i=1, \dots, m$, and $x, \bar{x} \in X$. Applying (8), hypothesis (v), (15), (10) and (14) we have

$$\begin{aligned} & \varrho_i(T_i(\varphi_1, \dots, \varphi_m)(x), T_i(\varphi_1, \dots, \varphi_m)(\bar{x})) \equiv \\ & \equiv a_i \varrho(x, \bar{x}) + \sum_{j=1}^m \sum_{k=1}^n a_{i,j,k} \varrho_j(\varphi_j[f_k(x)], \varphi_j[f_k(\bar{x})]) \equiv \\ & \equiv \left(a_i + \sum_{j=1}^m c_{i,j} l_j \right) \varrho(x, \bar{x}) \equiv l_i \varrho(x, \bar{x}), \end{aligned}$$

which shows that $T_i(\varphi_1, \dots, \varphi_m)$, $i=1, \dots, m$, fulfil condition (15), i.e. (16) holds. Moreover, condition (9) is fulfilled, where $\mathcal{C}_i$ and d_i , $i=1, \dots, m$, are defined as in the proof of theorem 1. By Matkowski's theorem the unique continuous solution $\varphi_i: X \rightarrow Y_i$, $i=1, \dots, m$, of system (1) must belong to $\mathcal{L}_i$, $i=1, \dots, m$, so it fulfils a Lipschitz condition.

Recalling once more the method of the proof of theorem 2 in [1] we obtain

Theorem 4. *If hypotheses (iii)—(v) and conditions (5) and (13) are fulfilled, where the constants $b_{\lambda,\mu}^\alpha$ and $c_{\lambda,\mu}^\alpha$, $\alpha=1, \dots, m$; $\lambda, \mu=1, \dots, m+1-\alpha$, are defined by (2)—(4) and (10)—(12), respectively, then system (1) has exactly one continuous solution $\varphi_i: X \rightarrow Y_i$, $i=1, \dots, m$. This solution fulfils a Lipschitz condition.*

3. In this section we shall give a theorem on the continuous dependence of continuous solution of the system (1) on the given functions. To this end consider a sequence of the systems of functional equations

$$(17) \quad \varphi_i(x) = h_{i,v}(x; \varphi_1[f_{1,v}(x)], \dots, \varphi_1[f_{n,v}(x)]; \dots; \varphi_m[f_{1,v}(x)], \dots, \varphi_m[f_{n,v}(x)]), \\ i = 1, \dots, m; \quad v = 0, 1, 2, \dots$$

and assume that

(vi) $h_{i,v}: X \times Y_1^n \times \dots \times Y_m^n \rightarrow Y_i$, $i = 1, \dots, m$, and $f_{k,v}: X \rightarrow X$, $k = 1, \dots, n$; $v = 0, 1, 2, \dots$, are continuous functions. Furthermore

$$\begin{aligned} & \varrho_i(h_{i,v}(x; y_{1,1}, \dots, y_{1,n}; \dots; y_{m,1}, \dots, y_{m,n}), h_{i,v}(x; \bar{y}_{1,1}, \dots, \bar{y}_{1,n}; \dots; \bar{y}_{m,1}, \dots, \bar{y}_{m,n})) \equiv \\ & \equiv \sum_{j=1}^m \sum_{k=1}^n a_{i,j,k} \varrho_j(y_{j,k}, \bar{y}_{j,k}), \end{aligned}$$

for every $x \in X$ and $y_{j,k}, \bar{y}_{j,k} \in Y_j$; $i, j = 1, \dots, m$; $k = 1, \dots, n$; $v = 0, 1, 2, \dots$, where $a_{i,j,k}$; $i, j = 1, \dots, m$; $k = 1, \dots, n$, are positive constants;

(vii) The sequences $\{h_{i,v}\}_{v=1}^{\infty}$ and $\{f_{k,v}\}_{v=1}^{\infty}$ tend uniformly on every compact set to $h_{i,0}$ and $f_{k,0}$, $i = 1, \dots, m$; $k = 1, \dots, n$, respectively.

In the proof of the theorem on the continuous dependence of the continuous solutions of system (1) we shall use the following

Lemma. Let (X_i, σ_i) , $i = 1, \dots, m$, be a complete metric spaces and suppose that the transformations $F_{i,v}: X_1 \times \dots \times X_m \rightarrow X_i$, $i = 1, \dots, m$; $v = 0, 1, 2, \dots$, fulfil

$$(18) \quad \sigma_i(F_{i,v}(x_1, \dots, x_m), F_{i,v}(\bar{x}_1, \dots, \bar{x}_m)) \leq \sum_{j=1}^m b_{i,j} \sigma_j(x_j, \bar{x}_j),$$

$$x_j, \bar{x}_j \in X_j; \quad i, j = 1, \dots, m; \quad v = 0, 1, 2, \dots,$$

with positive constants $b_{i,j}$; $i, j = 1, \dots, m$, and

$$(19) \quad F_{i,0}(x_1, \dots, x_m) = \lim_{v \rightarrow \infty} F_{i,v}(x_1, \dots, x_m), \quad x_j \in X_j; \quad i, j = 1, \dots, m.$$

If the constants $b_{\lambda,\mu}^\chi$, $\chi = 1, \dots, m$; $\lambda, \mu = 1, \dots, m+1-\chi$, defined by (3) and (4) fulfil (5), then the system

$$(20) \quad x_i = F_{i,v}(x_1, \dots, x_m), \quad i = 1, \dots, m; \quad v = 0, 1, 2, \dots,$$

has for every $v = 0, 1, 2, \dots$ exactly one solution $x_{i,v} \in X_i$, $i = 1, \dots, m$. This solution is given by

$$(21) \quad x_{i,v} = \lim_{\tau \rightarrow \infty} x_{i,v,\tau}, \quad i = 1, \dots, m; \quad v = 0, 1, 2, \dots$$

and

$$(22) \quad x_{i,v,\tau+1} = F_{i,v}(x_{1,v,\tau}, \dots, x_{m,v,\tau}), \quad i = 1, \dots, m; \quad v, \tau = 0, 1, 2, \dots,$$

where $x_{i,v,0}$ is an arbitrary element of X_i , $i = 1, \dots, m$; $v = 0, 1, 2, \dots$. Moreover,

$$(23) \quad x_{i,0} = \lim_{v \rightarrow \infty} x_{i,v}, \quad i = 1, \dots, m.$$

PROOF. The existence and uniqueness of solution $x_{i,v}$, $i = 1, \dots, m$; $v = 0, 1, 2, \dots$ of system (20) and formula (21) follows from Matkowski's theorem [3]. We shall show that (23) holds. Take $x_i \in X_i$, $i = 1, \dots, m$, and put $x_{i,v,0} = x_i$ for every $i = 1, \dots, m$ and $v = 0, 1, 2, \dots$. Next, by (3)–(5), (22) and (19) we may choose a system of positive numbers $r_1, \dots, r_m$ and a $\vartheta \in (0, 1)$ such that

$$(24) \quad \sum_{j=1}^m b_{i,j} r_j \leq \vartheta r_i, \quad i = 1, \dots, m,$$

and

$$\sigma_i(x_{i,v,1}, x_{i,v,0}) \leq r_i, \quad i = 1, \dots, m; \quad v = 0, 1, 2, \dots$$

(cf. [3], Lemma). By induction, applying (22), (18) and (24) we get

$$\sigma_i(x_{i,v,\tau+1}, x_{i,v,\tau}) \leq \vartheta^\tau r_i, \quad i = 1, \dots, m; \quad v, \tau = 0, 1, 2, \dots$$

This fact, jointly with (21), shows that the sequence

$$\{\sigma_i(x_{i,v}, x_{i,0})\}_{v=1}^{\infty}, \quad i = 1, \dots, m,$$

is bounded. Let us put

$$(25) \quad u_{i,v} = \sigma_i(x_{i,v}, x_{i,0}), \quad v_{i,v} = \sigma_i(F_{i,v}(x_{1,0}, \dots, x_{m,0}), x_{i,0}), \\ i = 1, \dots, m; \quad v = 1, 2, \dots,$$

$$(26) \quad v_{i,v,1} = v_{i,v}, \quad v_{i,v,\tau+1} = \sum_{j=1}^m b_{i,j} v_{j,v,\tau} + v_{i,v}, \quad i = 1, \dots, m; \quad v, \tau = 1, 2, \dots$$

Since the sequence $\{u_{i,v}\}_{v=1}^{\infty}$, $i = 1, \dots, m$, is bounded, we may require that the numbers r_i , $i = 1, \dots, m$, satisfying (24) fulfil also

$$(27) \quad u_{i,v} \leq r_i, \quad i = 1, \dots, m; \quad v = 1, 2, \dots$$

Taking into account (20), (18), (25), (19) and (26) we obtain

$$(28) \quad u_{i,v} \leq \sum_{j=1}^m b_{i,j} u_{j,v} + v_{i,v}, \quad i = 1, \dots, m; \quad v = 1, 2, \dots$$

and

$$(29) \quad \lim_{v \rightarrow \infty} v_{i,v,\tau} = 0, \quad i = 1, \dots, m; \quad \tau = 1, 2, \dots$$

Recalling (28), (27), (24), (26) and the induction principle we have

$$u_{i,v} \leq 9^v r_i + v_{i,v,\tau}, \quad i = 1, \dots, m; \quad v, \tau = 1, 2, \dots,$$

thus by (29)

$$\lim_{v \rightarrow \infty} u_{i,v} = 0, \quad i = 1, \dots, m,$$

i.e. (23) holds.

Now, we shall prove a theorem on the continuous dependence of the continuous solutions of system (1).

Theorem 5. *Let hypotheses (iv), (vi) and (vii) be fulfilled and suppose that X is a compact space. If the constants $b_{\lambda,\mu}^{\alpha}$, $\alpha = 1, \dots, m$; $\lambda, \mu = 1, \dots, m+1-\alpha$, defined by (2)–(4) fulfil condition (5), then system (17) has for every $v = 0, 1, 2, \dots$ exactly one continuous solution $\varphi_{i,v}: X \rightarrow Y_i$, $i = 1, \dots, m$, and the sequence $\{\varphi_{i,v}\}_{v=1}^{\infty}$ tends to $\varphi_{i,0}$, $i = 1, \dots, m$, uniformly in X .*

PROOF. Let $(\mathcal{C}_i, d_i)$, $i = 1, \dots, m$, be defined as in the proof of theorem 1, and put

$$T_{i,v}(\varphi_1, \dots, \varphi_m)(x) = \\ = h_{i,v}(x; \varphi_1[f_{1,v}(x)], \dots, \varphi_1[f_{n,v}(x)]; \dots; \varphi_m[f_{1,v}(x)], \dots, \varphi_m[f_{n,v}(x)]), \\ \varphi_j \in \mathcal{C}_j; \quad i, j = 1, \dots, m; \quad v = 0, 1, 2, \dots; \quad x \in X.$$

It follows from hypothesis (vi) that

$$T_{i,v}(\mathcal{C}_1 \times \dots \times \mathcal{C}_m) \subset \mathcal{C}_i, \quad i = 1, \dots, m; \quad v = 0, 1, 2, \dots$$

and

$$d_i(T_{i,v}(\varphi_1, \dots, \varphi_m), T_{i,v}(\bar{\varphi}_1, \dots, \bar{\varphi}_m)) \leq \sum_{j=1}^m b_{i,j} d_j(\varphi_j, \bar{\varphi}_j),$$

$$\varphi_j, \bar{\varphi}_j \in \mathcal{C}_j; \quad i, j = 1, \dots, m; \quad v = 0, 1, 2, \dots,$$

where $b_{i,j}$; $i, j = 1, \dots, m$, are defined by (2). Moreover,

$$\begin{aligned} & d_i(T_{i,v}(\varphi_1, \dots, \varphi_m), T_{i,0}(\varphi_1, \dots, \varphi_m)) \leq \\ & \leq \sup_{x \in X} \varrho_i(h_{i,v}(x; \varphi_1[f_{1,v}(x)], \dots, \varphi_1[f_{n,v}(x)]; \dots; \varphi_m[f_{1,v}(x)], \dots, \varphi_m[f_{n,v}(x)]), \\ & \quad h_{i,v}(x; \varphi_1[f_{1,0}(x)], \dots, \varphi_1[f_{n,0}(x)]; \dots; \varphi_m[f_{1,0}(x)], \dots, \varphi_m[f_{n,0}(x)])) + \\ & + \sup_{x \in X} \varrho_i(h_{i,v}(x; \varphi_1[f_{1,0}(x)], \dots, \varphi_1[f_{n,0}(x)]; \dots; \varphi_m[f_{1,0}(x)], \dots, \varphi_m[f_{n,0}(x)]), \\ & \quad h_{i,0}(x; \varphi_1[f_{1,0}(x)], \dots, \varphi_1[f_{n,0}(x)]; \dots; \varphi_m[f_{1,0}(x)], \dots, \varphi_m[f_{n,0}(x)])) \end{aligned}$$

for every $\varphi_j \in \mathcal{C}_j$; $i, j = 1, \dots, m$; $v = 1, 2, \dots$, so in view of hypotheses (vi), (vii) and of the compactness of X

$$T_{i,0}(\varphi_1, \dots, \varphi_m) = \lim_{v \rightarrow \infty} T_{i,v}(\varphi_1, \dots, \varphi_m), \quad \varphi_j \in \mathcal{C}_j; \quad i, j = 1, \dots, m.$$

Taking into account these facts and applying the above lemma we obtain our assertion.

It turns out that instead of the compactness of X we may assume that

(viii) There exists a sequence $\{G_\tau\}$ of open sets such that $X = \bigcup \{G_\tau; \tau = 1, 2, \dots\}$, $G_\tau \subset G_{\tau+1}$ and $\bar{G}_\tau$ is compact, $\tau = 1, 2, \dots$. Moreover, $f_{k,v}(G_\tau) \subset G_\tau$ for $k = 1, \dots, n$; every $v = 0, 1, 2, \dots$ and $\tau = 1, 2, \dots$.

Theorem 6. If hypotheses (iv), (vi)—(viii) and condition (5) are fulfilled, where the constants $b_{\lambda,\mu}^\alpha$, $\alpha = 1, \dots, m$; $\lambda, \mu = 1, \dots, m+1-\alpha$, are defined by (2)—(4), then system (17) has for every $v = 0, 1, 2, \dots$ exactly one continuous solution $\varphi_{i,v}: X \rightarrow Y_i$, $i = 1, \dots, m$, and the sequence $\{\varphi_{i,v}\}_{v=1}^\infty$ tends to $\varphi_{i,0}$, $i = 1, \dots, m$, uniformly on compact subset of X .

This theorem results from theorems 2 and 5, since every compact subset of X is contained in a G_τ .

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