

On logarithmic summability of conjugate series of Fourier series

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1. As in [1], we say that an infinite sequence $\{S_n\}$ is summable by the logarithmic method of summability or summable (L) to the sum s if, for x in the interval $(0, 1)$.

$$\lim_{x \rightarrow 1-0} \frac{1}{|\log(1-x)|} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n = s,$$

which is written simply as $S_n \rightarrow s(L)$.

In a paper [2], HSIANG has applied this method of summability to Fourier series of $f(x)$ and he obtained a summability criterion for it. The object of this paper is to make further application of this method in the theory of conjugate series of Fourier series.

2. Let $f(x)$ be a periodic function with period 2π , which is integrable in the sense of Lebesgue over the interval $(-\pi, \pi)$. Let the Fourier series of $f(x)$ be

$$(2.1) \quad \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(x).$$

Then the conjugate series of (2.1) is.

$$(2.2) \quad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx).$$

Fixing Θ , we write through out

$$\varphi(t) = \frac{1}{2} \{f(\Theta + t) + f(\Theta - t) - 2s\},$$

$$\psi(t) = \frac{1}{2} \{f(\Theta + t) - f(\Theta - t)\}.$$

3. HSIANG [2] proved the following theorem:

If

$$(3.1) \quad \int_0^t |\varphi(u)| du = o(t |\log t|), \quad (t \rightarrow +0),$$

$$(3.2) \quad \int_t^\delta (|\varphi(u)|/u) du = o(|\log t|),$$

as $t \rightarrow +0$ for any arbitrary $0 < \delta < \pi$, then the Fourier series of $f(x)$ is summable (L) to s at Θ .

We prove the following theorem:

Theorem: *If*

$$(3.3) \quad \int_0^t (|\psi(u)|/u) du = o(1), \quad (t \rightarrow +0),$$

$$(3.4) \quad \int_t^\delta (|\psi(u)|/u) du = o(|\log t|),$$

as $t \rightarrow +0$ for any arbitrary $0 < \delta < \frac{\pi}{2}$, then the conjugate series of Fourier series of $f(x)$ is summable (L) to

$$\frac{1}{\pi} \int_0^\pi \psi(t) \cot(t/2) dt$$

at $x = \Theta$, provided the sum integral exists in the Cauchy's sense.

4. PROOF OF THE THEOREM. Let

$$\tilde{S}_n(\Theta) = \sum_{v=1}^n (b_v \cos v\Theta - a_v \sin v\Theta)$$

be the n -th partial sum of (2.2). This partial sum can be represented as a definite integral. Using Euler—Fourier formulae for the coefficients a_v and b_v we have

$$\begin{aligned} \tilde{S}_n(\Theta) &= \frac{1}{\pi} \sum_{v=1}^n \left(\cos v\Theta \int_{-\pi}^{\pi} f(t) \sin vt dt - \sin v\Theta \int_{-\pi}^{\pi} f(t) \cos vt dt \right) \\ (4.1) \quad &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{v=1}^n (\cos v\Theta \sin vt - \sin v\Theta \cos vt) \right) dt = \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left(\sum_{v=1}^n \sin v(t - \Theta) \right) dt. \quad (\text{see } \S 2.3, [4]). \end{aligned}$$

By putting $t - \Theta = u$, and a little simplification will give us a more suitable form of it.

$$\begin{aligned} \tilde{S}_n(\Theta) &= \frac{1}{\pi} \int_{-\pi-\Theta}^{\pi-\Theta} f(\Theta + u) \left(\sum_{v=1}^n \sin vu \right) du, \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\Theta + u) \left(\sum_{v=1}^n \sin vu \right) du. \end{aligned}$$

Breaking up the range of the integral, we have.

$$\begin{aligned} \tilde{S}_n(\Theta) &= \frac{1}{\pi} \left[\int_{-\pi}^0 f(\Theta + u) \left(\sum_{v=1}^n \sin vu \right) du + \int_0^{\pi} f(\Theta + u) \left(\sum_{v=1}^n \sin vu \right) du \right] \\ &= \frac{1}{\pi} \left[- \int_0^{\pi} f(\Theta - u) \left(\sum_{v=1}^n \sin vu \right) du + \int_0^{\pi} f(\Theta + u) \left(\sum_{v=1}^n \sin vu \right) du \right]. \end{aligned}$$

This is obtained by writing $u = -v$ in the range $(-\pi, 0)$ i.e. in the first integral. So we have

$$\begin{aligned} \tilde{S}_n(\Theta) &= \frac{1}{\pi} \int_0^\pi \{f(\Theta + u) - f(\Theta - u)\} \left(\sum_{v=1}^n \sin vu \right) du. \\ (4.2) \quad &= \frac{2}{\pi} \int_0^\pi \psi(t) \left(\sum_{v=1}^n \sin vt \right) dt. \end{aligned}$$

Since

$$\sum_{v=1}^n \sin vt = \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{2 \sin \frac{1}{2}t}$$

(for this sum the reader may refer § 1.12, [4]) we have

$$(4.3) \quad \tilde{S}_n(\Theta) = \frac{1}{\pi} \int_0^\pi \psi(t) \cdot \frac{\cos \frac{1}{2}t - \cos(n + \frac{1}{2})t}{\sin \frac{1}{2}t} dt.$$

Using (4.3), we have

$$\begin{aligned} \sum_{n=1}^\infty \tilde{S}_n(\Theta) \frac{x^n}{n} &= \sum_{n=1}^\infty \frac{x^n}{n} \left(\frac{1}{\pi} \int_0^\pi \psi(t) \frac{\cos(t/2) - \cos(n + \frac{1}{2})t}{\sin(t/2)} dt \right) = \\ &= \sum_{n=1}^\infty \frac{x^n}{n} \left(\frac{1}{\pi} \int_0^\pi \psi(t) \cot(t/2) dt \right) - \\ &- \sum_{n=1}^\infty \frac{x^n}{n} \left(\frac{1}{\pi} \int_0^\pi \psi(t) \frac{\cos(n + \frac{1}{2})t}{\sin(t/2)} dt \right). \end{aligned}$$

Let us put

$$\frac{1}{\pi} \int_0^\pi \psi(t) \cot(t/2) dt = s.$$

Then we have

$$\begin{aligned} \sum_{n=1}^\infty \{\tilde{S}_n(\Theta) - s\} \frac{x^n}{n} &= \frac{1}{\pi} \int_0^\pi \frac{\psi(t)}{\sin(t/2)} \left(- \sum_{n=1}^\infty \frac{x^n}{n} \cos\left(n + \frac{1}{2}\right)t \right) dt, \\ &= \frac{1}{\pi} \int_0^\pi \frac{\psi(t)}{\sin(t/2)} K(t, x) dt, \end{aligned}$$

where $K(t, x) = \left(- \sum_{n=1}^\infty \frac{x^n}{n} \cos\left(n + \frac{1}{2}\right)t \right)$.

In order to prove the theorem, we have to show that

$$(4.4) \quad \int_0^\pi \frac{\psi(t)}{\sin(t/2)} K(t, x) dt = o(|\log(1-x)|) \quad \text{as } x \rightarrow 1-0.$$

Using the common method of summation which is known as “ $c+is$ method” we

get the sum of the series as follows: Let

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \cos \left(n + \frac{1}{2} \right) t = c \quad (\text{say}).$$

Then we have

$$C = x \cos \frac{3}{2} t + \frac{x^2}{2} \cos \frac{5}{2} t + \dots + \frac{x^n}{n} \cos \left(n + \frac{1}{2} \right) t + \dots$$

The corresponding auxiliary sine series would be

$$S = x \sin \frac{3}{2} t + \frac{x^2}{2} \sin \frac{5}{2} t + \dots + \frac{x^n}{n} \sin \left(n + \frac{1}{2} \right) t + \dots$$

Then we have

$$\begin{aligned} C + iS &= x \left(\cos \frac{3}{2} t + i \sin \frac{3}{2} t \right) + \frac{x^2}{2} \left(\cos \frac{5}{2} t + i \sin \frac{5}{2} t \right) + \dots \\ &\quad \dots + \frac{x^n}{n} \left(\cos \left(n + \frac{1}{2} \right) t + i \sin \left(n + \frac{1}{2} \right) t \right) + \dots = \\ &= x e^{i \frac{3}{2} t} + \frac{x^2}{2} e^{i \frac{5}{2} t} + \dots + \frac{x^n}{n} e^{i \left(n + \frac{1}{2} \right) t} + \dots = \\ &= e^{it/2} \left(x e^{it} + \frac{x^2}{2} e^{2it} + \dots + \frac{x^n}{n} e^{nit} + \dots \right) = \\ &= -e^{it/2} \log_e (1 - x e^{it}) = -e^{it/2} \log_e (1 - x \cos t - ix \sin t). \end{aligned}$$

We know that general value of logarithm of a complex quantity $(\alpha + \beta i)$ is given by the formula.

$$\text{Log}_e(\alpha + \beta i) = \log_e(\alpha^2 + \beta^2)^{\frac{1}{2}} + i(2n\pi + \text{arctg } \beta/\alpha)$$

and its principal value

$$\log_e(\alpha + \beta i) = \log_e(\alpha^2 + \beta^2)^{\frac{1}{2}} + i \text{arctg } (\beta/\alpha).$$

so we have

$$\begin{aligned} C + iS &= -e^{it/2} \left[\log_e \{ (1 - x \cos t)^2 + (-x \sin t)^2 \}^{\frac{1}{2}} + i \text{arctg } \frac{-x \sin t}{1 - x \cos t} \right] \\ &= -e^{it/2} \left[\log_e (1 - 2x \cos t + x^2)^{\frac{1}{2}} - i \text{arctg } \frac{x \sin t}{1 - x \cos t} \right] \\ &= - \left(\cos \frac{t}{2} + i \sin \frac{t}{2} \right) \left[\log_e (1 - 2x \cos t + x^2)^{\frac{1}{2}} - i \text{arctg } \frac{x \sin t}{1 - x \cos t} \right]. \end{aligned}$$

Comparing real and imaginary parts, we have

$$C = - \left[\frac{1}{2} \cos \frac{t}{2} \log_e (1 - 2x \cos t + x^2) + \sin \frac{t}{2} \text{arctg } \frac{x \sin t}{1 - x \cos t} \right].$$

Hence we have

$$\begin{aligned} K(t, x) &= - \sum_{n=1}^{\infty} \frac{x^n}{n} \cos \left(n + \frac{1}{2} \right) t = \\ &= \frac{1}{2} \cos (t/2) \log_e (1 - 2x \cos t + x^2) + \sin (t/2) \operatorname{arctg} \frac{x \sin t}{1 - x \cos t}. \end{aligned}$$

Hence

$$\begin{aligned} &\int_0^{\pi} \frac{\psi(t)}{\sin(t/2)} k(t, x) dt = \\ &= \int_0^{\pi} \frac{\psi(t)}{\sin(t/2)} \left\{ \frac{1}{2} \cos(t/2) \log_e (1 - 2x \cos t + x^2) \right\} + \\ &+ \int_0^{\pi} \psi(t) \operatorname{arctg} (x \sin t / 1 - x \cos t) dt = \\ &= I_1 + I_2 \quad (\text{say}). \end{aligned}$$

Since the period of $\operatorname{tg} \alpha$ is π , we can consider

$$|\operatorname{arctg} (x \sin t / 1 - x \cos t)| < \pi/2$$

uniformly for $0 \leq x < 1$ and $0 \leq t \leq \pi$, we find that $I_2 = O(1) = o(|\log_e(1-x)|)$, as $x \rightarrow 1-0$. Now we proceed to show that under the given conditions $I_1 = o(|\log_e(1-x)|)$ as $x \rightarrow 1-0$.

We break up the integral I_1 as follows:

$$\begin{aligned} I_1 &= \int_0^{1-x} + \int_{1-x}^{\delta} + \int_{\delta}^{\pi} \\ &= I_{11} + I_{12} + I_{13}. \end{aligned}$$

We have by (3.3) and (3.4)

$$\begin{aligned} I_{11} &= \int_0^{1-x} \frac{\psi(t)}{\sin(t/2)} \left\{ \frac{1}{2} \cos(t/2) \log_e (1 - 2x \cos t + x^2) \right\} dt = \\ &= O \left(\int_0^{1-x} \frac{|\psi(t)|}{t} |\log_e (1 - 2x \cos t + x^2)| dt \right) = \\ &= O \left(|\log_e (1 - 2x \cos (1-x) + x^2)| \int_0^{1-x} \frac{|\psi(t)|}{t} dt \right) = \\ &= o(|\log_e(1-x)|), \quad \text{as } x \rightarrow 1-0; \\ I_{12} &= \int_{1-x}^{\delta} \frac{\psi(t)}{\sin(t/2)} \left\{ \frac{1}{2} \cos(t/2) \log_e (1 - 2x \cos t + x^2) \right\} dt = \\ &= O \left(\int_{1-x}^{\delta} \frac{|\psi(t)|}{|\sin(t/2)|} |\log_e (1 - 2x \cos t + x^2)| dt \right). \end{aligned}$$

Since $\frac{\sin t}{t}$ is steadily decreasing for $0 < t < \frac{\pi}{2}$, we have $\frac{1}{\sin(t/2)} \cong \frac{\pi}{t}$, (page 7, [3]); and also $|\log_e(1 - 2x \cos t + x^2)|$ attains its maximum at the upper limit of the integral $t = \delta$. We have $|\log_e(1 - 2x \cos t + x^2)| = O(1)$, as $x \rightarrow 1-0$, for the integral under consideration. Therefore, the estimate of I_{12} would be

$$\begin{aligned} I_{12} &= O \left(\int_{1-x}^{\delta} \frac{|\psi(t)|}{t} dt \right) = \\ &= o(|\log_e(1-x)|), \quad \text{as } x \rightarrow 1-0; \end{aligned}$$

Finally, we have

$$\begin{aligned} I_{13} &= O \left(|\log_e(1-x)| \int_{\delta}^{\pi} \psi(t) dt \right) = \\ &= O(1) = o(|\log_e(1-x)|), \quad \text{as } x \rightarrow 1-0. \end{aligned}$$

This completes the proof of the theorem.

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